3E1 Hilary Term Tutorial sheet 5
[February 19–24, 2003]

Name: Solution

1. The circularly symmetric fundamental (or modal) solutions to the wave equation for a circular drum of radius a are given by

$$u_n(r,t) = \left(A_n \cos\left(\frac{ca_{0n}}{a}t\right) + B_n \sin\left(\frac{ca_{0n}}{a}t\right)\right) J_0\left(\frac{a_{0n}}{a}r\right)$$

for the Bessel function J_0 and the positive zeros $0 < a_{01} < a_{02} < a_{03} < \cdots$ of J_0.

Find the nodal lines (in fact nodal circles) for u_n. (These are the points where $u_n(r,t) = 0$ for all t.)

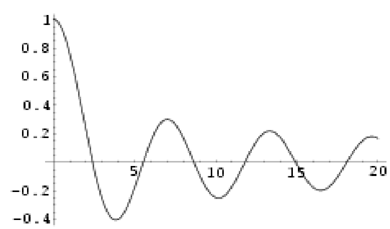
Solution: We seek the values of r where $J_0\left(\frac{a_{0n}}{a}r\right) = 0$ and this means that

$$\begin{aligned} \frac{a_{0n}}{a}r &= a_{0m} \text{ for some (maybe other) } m \\ r &= \frac{a_{0m}}{a_{0n}}a \end{aligned}$$

Since r is restricted to $0 \leq r \leq a$ we have to have $a_{0m} \leq a_{0m}$ or $m \leq n$. The case $m = n$ only gives the outer rim of the drum (which is fixed anyhow by the boundary condition) and so the nodal circles are

$$r = \frac{a_{0m}}{a_{0n}}a \quad m = 1, 2, \ldots, n-1$$

A graph of the Bessel function J_0 and a picture (see Fig. 1) of a snapshot of u_3 at a time when the origin is depressed down. (The picture has a sector cut out and has a horizontal cross through the origin). The horizontal cuts the surface at two intermediate circles.



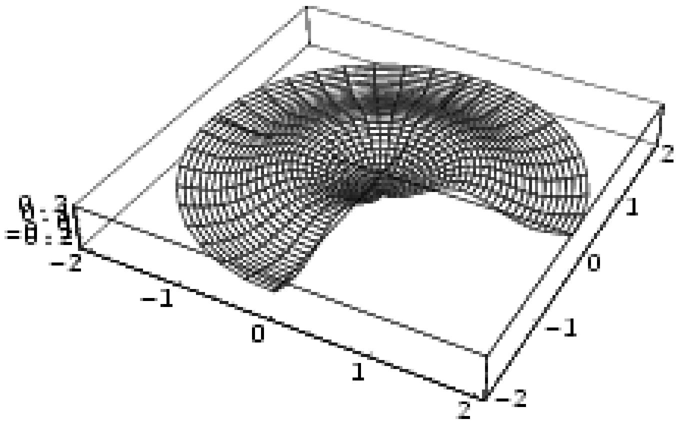


Figure 1: Snapshot of u_3

2 A thin circular disk of radius 2 is insulated on its faces. The round edge is kept at temperature 0 and the temperature $u(x, y, t)$ (at points (x, y) of the disk) obeys the heat equation

$$\frac{\partial u}{\partial t} = 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Find the circularly symmetric product form solutions (using polar coordinates).

Solution: Here we proceed in a way similar to the circular drum, but there are some differences because of the first order partial with respect to t.

Let (in polar coordinates) $u(r, \theta, t) = u(r, t) = F(r)G(t)$ (no θ dependence since we are dealing with the circularly symmetric case). Then $\frac{\partial u}{\partial t} = F(r)G'(t)$ and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = F''(r)G(t) + \frac{1}{r}F'(r)G(t)$$

so that the PDE says

$$\begin{aligned} F(r)G'(t) &= 2\left(F''(r)G(t) + \frac{1}{r}F'(r)G(t)\right) \\ \frac{G'(t)}{2G(t)} &= \frac{F''(r)}{F(r)} + \frac{1}{r}\frac{F'(r)}{F(r)} \end{aligned}$$

(on dividing across by $2F(r)G(t)$). So both sides have to be constant, say $= k$.

Now $G'(t) - 2kG(t) = 0$ and this has solutions $G(t) = Ae^{2kt}$ (for A a constant). If $k > 0$ then this means that the absolute value $|G(t)| \to \infty$ as $t \to \infty$ and this seems to be unlikely on physical grounds (that $|u(r, t)| \to \infty$ as $t \to \infty$ for fixed r). To rule it out in a mathematical way is also possible, by considering the ODE for F and the boundary conditions, but this would take quite a bit of extra work. For $k = 0$ we have $G(t) = A =$ constant, or a steady state solution. This is not ruled out on physical grounds. So we have $k = -p^2 \leq 0$ with $p \geq 0$.

The ODE for $F(r)$ is then

$$F''(r) + \frac{1}{r}F'(r) + p^2 F(r) = 0$$

For $p > 0$ we know the solution that is finite at $r = 0$ is $F(r) = J_0(pr)$ (and multiples of that).

We deduced this by the change of variables $s = pr$, $r = s/p$. Let $h(s) = F(s/p)$. Now $h'(s) = \frac{1}{p}F'(s/p)$, $h''(s) = \frac{1}{p^2}F''(s/p)$. The ODE becomes $p^2 h''(s) + \frac{p}{s}ph'(s) + p^2 h(s) = 0$. Dividing by p^2 we get the ODE for J_0. The other solutions of Bessels equation have a logarithmic term (involve $\ln x$) and so are infinite at 0. So $h(s) = J_0(s)$ (or a multiple).

The boundary condition $F(2) = 0$ tells us that $J_0(2p) = 0$ and so $2p$ is one of the zeros $0 < a_{01} < a_{02} < \cdots$ of J_0. $2p = a_{0n}$ (say) or $p = a_{0n}/2$. This leads us to the modal solutions

$$u_n(r,t) = e^{-2p^2 t} J_0(pr) = \exp(-(a_{0n}^2/2)t) J_0\left(a_{0n} r/2\right) \quad n = 1, 2, \ldots$$

(or a constant A_n times that).

We should still look into the case $p = 0$. Then the ODE for F is $F''(r) + (1/r)F'(r) = 0$ which is a first order linear equation for $F'(r)$. If we put $H(r) = F'(r)$ it is $H'(r) + (1/r)H(r) = 0$ and this kind of ODE can be solved by the integrating factor method. (First order linear ODE. Integrating factor $\exp\left(\int (1/r)\,dr\right) = \exp \ln r = r$.) We get $rH'(r) + H(r) = 0$ or $\frac{d}{dr}(rH(r)) = 0$. Thus $rH(r) = B =$ constant, $F'(r) = H(r) = B/r$ and so $F(r) = B \ln r + C$. We cannot have the log term as $F(0)$ would not be defined. So $B = 0$ and $F(r) = C =$ constant, but the boundary condition $F(2) = 0$ then forces $C = 0$. So we end up with only the zero solution if $k = 0$. So we already had all the fundamental solutions u_n above.