**3E1 Hilary Term Tutorial sheet 4**
[February 12–17, 2003]

**Name:** Solutions

1. For a thin elastic vibrating string of length 2 fixed at both ends the vertical displacement $u(x,t)$ obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0 \le x \le 2$$

where $c = 4$. We know that the solutions $u$ may be represented by series $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ where $u_n(x,t) = (A_n \cos((cn\pi/2)t) + B_n \sin((cn\pi/2)t)) \sin((n\pi/2)x)$. If the string is stretched at time $t = 0$ so that $u(x,0) = 0.1(2x - x^2)$ and let go from rest, find $u$ (as a series).

*Solution:* Letting go from rest means $\partial u/\partial t = 0$ at $t = 0$. So

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (cn\pi/2)(-A_n \sin((cn\pi/2)t) + B_n \cos((cn\pi/2)t)) \sin((n\pi/2)x)$$

is 0 at $t = 0$. Thus $\sum_{n=1}^{\infty} (cn\pi/2) B_n \sin((n\pi/2)x) = 0$ and we conclude $B_n = 0$ for all $n$. Then from $u(x,0) = \sum_{n=1}^{\infty} A_n \sin((n\pi/2)x) = 0.1(2x - x^2)$ we see that the $A_n$ must be the Fourier sine series coefficients of $0.1(2x - x^2)$, or

$$\begin{aligned}
A_n &= \frac{2}{2} \int_0^2 0.1(2x - x^2) \sin((n\pi/2)x)\, dx \\
&\quad \text{Use integration by parts} \\
&\quad u = 0.1(2x - x^2), \quad dv = \sin((n\pi/2)x)\, dx \\
&\quad du = 0.2(1 - x)\, dx, \quad v = -\frac{2}{n\pi} \cos((n\pi/2)x) \\
&= \int_0^2 u\, dv \\
&= [uv]_0^2 - \int_0^2 v\, du \\
&= \left[ -0.1(2x - x^2) \frac{2}{n\pi} \cos((n\pi/2)x) \right]_0^2 + \int_0^2 \frac{2}{n\pi} \cos((n\pi/2)x) 0.2(1 - x)\, dx \\
&= \frac{0.4}{n\pi} \int_0^2 (1 - x) \cos((n\pi/2)x)\, dx \\
&\quad \text{Use integration by parts again} \\
&\quad U = 1 - x, \quad dV = \cos((n\pi/2)x)\, dx \\
&\quad dU = -dx, \quad V = \frac{2}{n\pi} \sin((n\pi/2)x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{0.4}{n\pi}\left([UV]_0^2 - \int_0^2 V\,dU\right)\\
&= \frac{0.4}{n\pi}\left(\left[\frac{2}{n\pi}(1-x)\sin((n\pi/2)x)\right]_0^2 + \int_0^2 \frac{2}{n\pi}\sin((n\pi/2)x)\,dx\right)\\
&= \frac{0.8}{n^2\pi^2}\int_0^2 \sin((n\pi/2)x)\,dx\\
&= \frac{0.8}{n^2\pi^2}\left[-\frac{2}{n\pi}\cos((n\pi/2)x)\right]_0^2\\
&= \frac{1.6}{n^3\pi^3}(-\cos(n\pi)+\cos 0)\\
&= \frac{1.6}{n^3\pi^3}(1-(-1)^n)
\end{aligned}$$

So $A_n = 0$ for $n$ even and for $n$ odd $A_n = 3.2/(n^3\pi^3)$.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{3.2}{(2n-1)^3\pi^3}\cos(c(n-1/2)\pi t)\sin((n-1/2)\pi x)$$

(with $c = 4$).

2. For a thin elastic membrane in the shape of a rectangle $0 \le x \le 2$, $0 \le y \le 1$ the vertical displacement $u(x,y,t)$ obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

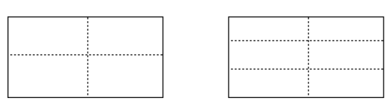
with $c = 4$. The solutions $u(x,y,t)$ are superpositions (infinite series) of terms

$$u_{nm}(x,y,t) = (A_{nm}\cos\lambda_{nm}t + B_{nm}\sin\lambda_{nm}t)\sin\frac{n\pi x}{2}\sin m\pi y$$

where $\lambda_{nm} = c\pi\sqrt{n^2/4+m^2}$. For the two cases $n = 2$ and $m = 2, 3$ sketch the nodal lines (lines in the $x$-$y$ plane where $u_{nm}(x,y,t) \equiv 0$ for all $t$). Also find the frequencies for the two $u_{nm}$.

*Solution:* The nodal lines happen where $\sin\frac{n\pi x}{2}\sin m\pi y = 0$, which means $\frac{n\pi x}{2}$ an integer multiple of $\pi$ or $m\pi y$ a multiple of $\pi$. With $n = 2$, $m = 2$ this means $\pi x$ a multiple of $\pi$ or $2\pi y$ a multiple of $\pi$. So $x = 1$ or $y = 1/2$ are the ones inside the region.

For $n = 2$, $m = 3$ we have $x = 1$ and $y = 1/3$, $y = 2/3$.



$u_{nm}$ has period $2\pi/\lambda_{nm}$ and so frequency $\lambda_{nm}/(2\pi) = (c/2)\sqrt{n^2/4+m^2}$. For $(n,m) = (2,2)$ the frequency is $2\sqrt{5}$ and for $(n,m) = (2,3)$ it is $2\sqrt{10}$.