3E1 Hilary Term Tutorial sheet 3
[February 5–10, 2003]

Name: Solutions

1. A thin elastic rod vibrates longitudinally. The displacement $u(x,t)$ at time t of the part of the rod initially at x obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

We assume the extent of the rod is $0 \leq x \leq \ell$ and that the end at $x = 0$ is fixed (mathematically $u(0,t) \equiv 0$) while the other end is free (mathematically $\frac{\partial u}{\partial x}(\ell, t) \equiv 0$ corresponds to zero force on the end).

(a) Find the product solutions $u(x,t) = F(x)G(t)$ for this problem.
Solution:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= F(x)G''(t) \\ \frac{\partial^2 u}{\partial x^2} &= F''(x)G(t) \end{aligned}$$

so the the PDE becomes

$$\begin{aligned} F(x)G'(t) &= c^2 F''(x)G(t) \\ \frac{G'(t)}{G(t)} &= c^2 \frac{F''(x)}{F(x)} \end{aligned}$$

As the two sides involve different variables, the only way they can be equal is for both sides to be constant. Call the constant k.

$$\begin{aligned} c^2 \frac{F''(x)}{F(x)} &= k \\ F''(x) - \frac{k}{c^2} F(x) &= 0 \\ \left(D^2 - \frac{k}{c^2} \right) F(x) &= 0 \end{aligned}$$

If $k > 0$ the solutions are $F(x) = A_1 \exp((\sqrt{k}/c)x) + A_2 \exp(-(\sqrt{k}/c)x)$. Then the boundary conditions force

$$\begin{aligned} 0 = u(0,t) &= F(0)G(t) \\ &= (A_1 + A_2)G(t) \\ 0 = \frac{\partial u}{\partial x}(\ell, t) &= F'(\ell)G(t) \\ &= (A_1(\sqrt{k}/c) \exp((\sqrt{k}/c)\ell) - A_2(\sqrt{k}/c) \exp(-(\sqrt{k}/c)\ell)G(t) \end{aligned}$$

so that either $G(t) \equiv 0$ or we can solve two simultaneous equations for A_1, A_2:

$$\begin{bmatrix} 1 & 1 \\ (\sqrt{k}/c)\exp((\sqrt{k}/c)\ell) & -(\sqrt{k}/c)\exp(-(\sqrt{k}/c)\ell) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

The matrix has determinant $-(\sqrt{k}/c)(\exp(-(\sqrt{k}/c)\ell) + \exp((\sqrt{k}/c)\ell)) \neq 0$ and so $A_1 = A_2 = 0$. Either way we have only $u(x,t) \equiv 0$.

If $k = 0$, we have $F(x) = A_1 + A_2 x$ and we can similarly use the boundary conditions

$$\begin{aligned} 0 = u(0,t) &= F(0)G(t) = A_1 G(t) \\ 0 = \frac{\partial u}{\partial x}(\ell,t) = F'(\ell)G(t) &= A_2 G(t) \end{aligned}$$

and we end up with only the zero solution for $u(x,t)$ again.

If $k < 0$ we can write $-\frac{k}{c^2} = \tau^2$ (say) and then the solutions of $(D^2 + \tau^2)F(x) = 0$ are

$$F(x) = A_1 \cos \tau x + A_2 \sin \tau x$$

The boundary condtions tells us

$$\begin{aligned} 0 = u(0,t) = F(0)G(t) &= A_1 G(t) \\ \Rightarrow A_1 &= 0 \text{ assuming } G(t) \not\equiv 0 \\ 0 = \frac{\partial u}{\partial x}(\ell,t) = F'(\ell)G(t) &= A_2 \tau \cos(\tau \ell) G(t) \end{aligned}$$

Thus $\tau = (n + 1/2)\pi/\ell$ for some integer n (or $\tau = 0$ but this makes $F(x) \equiv 0$). Thus $u(x,t) = A_2 \sin(\tau x)G(t) = A_2 \sin(((n+1/2)\pi/\ell)x)G(t)$ and G solves

$$\begin{aligned} \frac{G''(t)}{G(t)} &= k = -\tau^2 c^2 \\ G''(t) + \tau^2 c^2 G(t) &= 0 \\ G(t) &= B_1 \cos \tau c t + B_2 \sin \tau c t \end{aligned}$$

So $u(x,t) = A_2 \sin(((n+1/2)\pi/\ell)x)(B_1 \cos(((n+1/2)c\pi/\ell)t) + B_2 \sin(((n+1/2)c\pi/\ell)t)$

(b) If the initial displacment (stretching) is given by $u(x,0) = \sin(3\pi x/(2\ell))$ and the rod is started from rest ($\dfrac{\partial u}{\partial t}(x,0) \equiv 0$), find a solution $u(x,t)$.

Solution: We should look for u of the form of a series

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n \sin\left(\frac{(n+1/2)\pi}{\ell}x\right)\left(B_n \cos\left(\frac{(n+1/2)c\pi}{\ell}t\right) + C_n \sin\left(\frac{(n+1/2)c\pi}{\ell}t\right)\right)$$

From $u(x,0) = \sin(3\pi x/(2\ell))$ we get $\sin(3\pi x/(2\ell)) = \sum_{n=-\infty}^{\infty} A_n B_n \sin\left(\frac{(n+1/2)\pi}{\ell}x\right)$ and so it works to have $A_1 B_1 = 1$ while all other $A_n B_n = 0$.

Taking the other condition we get $\frac{\partial u}{\partial t}(x,0) = \sum_{n=-\infty}^{\infty} A_n C_n \left(\frac{(n+1/2)c\pi}{\ell}\right) \sin\left(\frac{(n+1/2)\pi}{\ell}x\right) = 0$. Thus it works to have all $A_n C_n = 0$ and

$$u(x,t) = \sin\left(\frac{3\pi}{2\ell}x\right)\cos\left(\frac{3c\pi}{2\ell}t\right)$$