3E1 Hilary Term Tutorial sheet 2
[January 15–February 3, 2003]

Name: Solutions

1. Use the divergence theorem and problem 2 on the previous sheet to show that for $u(x,y,z) = 1/\sqrt{x^2+y^2+z^2}$ we have
$$\iint_S (\nabla u).\mathbf{n}\,dA = 0$$
if S is the outer surface of a filled-in region of space that does not contain the origin and $\mathbf{n}$ is the outward unit normal to S. Where do you need the fact that the origin is not inside or on S?

   Solution: From the divergence theorem
$$\begin{aligned}\iint_S (\nabla u).\mathbf{n}\,dA &= \iiint_R \nabla.(\nabla u)\,dx\,dy\,dz\\ &= \iiint_R \nabla^2 u\,dx\,dy\,dz\\ &= 0\end{aligned}$$
since u satisfies Laplace's equation $\nabla^2 u = 0$. Here R stands for the region of space inside S and it is important that the origin is not in R because u is not well behaved there. Hence we cannot necessarily apply the divergence theorem if the origin is in R (or on its surface S).

   Aside. If S is the unit sphere, you can check that $(\nabla u).\mathbf{n} = 1$ on S and so $\iint_S (\nabla u).\mathbf{n}\,dA = 2(4\pi/3) \neq 0$.

2. Find the product solutions $u(x,t) = F(x)G(t)$ for the PDE
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$
with $0 \leq x \leq \ell$ and boundary conditions $u(0,t) = 0$, $u(\ell,t) = 0$. [This is in fact a 1-dimensional heat equation for a thin bar with a perfectly insulating surrounding cover and ends kept at 0 temperature.]

   Solution: Write
$$\begin{aligned}\frac{\partial u}{\partial t} &= F(x)G'(t)\\ \frac{\partial u}{\partial x} &= F'(x)G(t)\\ \frac{\partial^2 u}{\partial x^2} &= F''(x)G(t)\end{aligned}$$

so the the PDE becomes

$$\begin{aligned} F(x)G'(t) &= c^2F''(x)G(t) \\ \frac{G'(t)}{G(t)} &= c^2\frac{F''(x)}{F(x)} \end{aligned}$$

As the two sides involve different variables, the only way they can be equal is for both sides to be constant. Call the constant k.

$$\begin{aligned} c^2\frac{F''(x)}{F(x)} &= k \\ F''(x) - \frac{k}{c^2}F(x) &= 0 \\ \left(D^2 - \frac{k}{c^2}\right)F(x) &= 0 \end{aligned}$$

If $k > 0$ the solutions are $F(x) = A_1\exp((\sqrt{k}/c)x) + A_2\exp(-(\sqrt{k}/c)x)$. Then the boundary conditions force

$$\begin{aligned} 0 = u(0,t) = F(0)G(t) &= (A_1 + A_2)G(t) \\ 0 = u(\ell,t) = F(\ell)G(t) &= (A_1\exp((\sqrt{k}/c)\ell) + A_2\exp(-(\sqrt{k}/c)\ell)G(t) \end{aligned}$$

so that either $G(t) \equiv 0$ or we can solve two simultaneous equations for A_1, A_2 to get $A_1 = A_2 = 0$. Either way we have only $u(x,t) \equiv 0$.

If $k = 0$, we have $F(x) = A_1 + A_2x$ and we can similarly use the boundary conditions

$$\begin{aligned} 0 = u(0,t) = F(0)G(t) &= A_1G(t) \\ 0 = u(\ell,t) = F(\ell)G(t) &= (A_1 + A_2\ell)G(t) \end{aligned}$$

and we end up with only the zero solution for $u(x,t)$ again.

If $k < 0$ we can write $-\frac{k}{c^2} = \tau^2$ (say) and then the solutions of $(D^2 + \tau^2)F(x) = 0$ are

$$F(x) = A_1\cos\tau x + A_2\sin\tau x$$

The boundary condtions tell us

$$\begin{aligned} 0 = u(0,t) = F(0)G(t) &= A_1G(t) \\ \Rightarrow A_1 &= 0 \text{ assuming } G(t) \not\equiv 0 \\ 0 = u(\ell,t) = F(\ell)G(t) &= A_2\sin\tau\ell G(t) \end{aligned}$$

Thus $\tau = n\pi/\ell$ for some integer n. (Replacing n by $-n$ makes no essential difference as $\sin(-n\pi x/\ell) = -\sin n\pi x/\ell$ only makes a sign change. $n = 0$ gives 0 also. So we can suppose $n > 0$ is a positive integer.) Thus $u(x,t) = B_2\sin(n\pi x/\ell)G(t)$ and G solves

$$\begin{aligned} \frac{G'(t)}{G(t)} &= k = -\tau^2c^2 = -n^2\pi^2c^2/\ell^2 \\ G'(t) - kG(t) &= 0 \\ G(t) &= Be^{kt} = B\exp(-\tau^2c^2t) = B\exp(-(n^2\pi^2c^2/\ell^2)t) \end{aligned}$$

So $u(x,t) = A_2B\sin((n\pi/\ell)x)\exp(-(n^2\pi^2c^2/\ell^2)t)$.