3E1 Hilary Term Tutorial sheet 1
[January 8–13, 2003]

Name: Solutions

1. Find the general solution of the PDE

$$\frac{\partial^2 u}{\partial y^2} = u \qquad (u = u(x, y)).$$

Solution: For fixed x we have an ODE satisfied by $f(y) = u(x, y)$, namely $f'' = f$ or $(D^2 - 1)f = 0$ or $(D - 1)(D + 1)f = 0$. Thus $f(y) = Ae^y + Be^{-y}$ for constants A, B. But the constants can change when we go to another x. So

$$u(x, y) = A(x)e^y + B(x)e^{-y}$$

with $A(x)$ and $B(x)$ arbitrary functions.

2. Show that $u(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$ solves the (3 dimensional) Laplace equation.

Solution: Laplace says

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u = 0$$

(or also written $\nabla^2 u = 0$ or $\triangle u = 0$). Now

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} + 3\frac{x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&\quad \text{(similarly for } y \text{ and } z \text{ partials)} \\
\nabla^2 u &= \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} + 3\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= 0.
\end{aligned}$$

Remark. *This is known as a fundamental solution of Lalace's equation (with pole at the origin). It corresponds to the electrostatic potential of a point charge at the origin.*

3. Show that if $f(x) = \sin\left(\frac{\pi}{\ell}x\right)$, then $u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct))$ solves the (1 dimensional) wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ together with boundary conditions $u(0, t) \equiv 0 \equiv u(\ell, t)$ and initial conditions $u(x, 0) = f(x)$, $\frac{\partial u}{\partial t}(x, 0) = 0$.

Solution: The particular form of the function $f(x)$ is not really needed for the whole problem and it may be easier not to use that until it is needed. Recall that we have a general solution of the Heat equation $u(x, t) = \phi(x + ct) + \psi(x - ct)$ and what are checking here is that something that fits that pattern is indeed a solution. [No surprise that it does.]

Using the chain rule, we can see

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}\left(f'(x+ct)\frac{\partial}{\partial x}(x+ct) + f'(x-ct)\frac{\partial}{\partial x}(x-ct)\right) \\
&= \frac{1}{2}(f'(x+ct) + f'(x-ct)) \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{2}(f''(x+ct) + f''(x-ct)) \\
\frac{\partial u}{\partial t} &= \frac{1}{2}(cf'(x+ct) - cf'(x-ct)) \\
\frac{\partial^2 u}{\partial t^2} &= \frac{1}{2}(c^2 f''(x+ct) + c^2 f''(x-ct)) \\
&= c^2\left(\frac{1}{2}(f''(x+ct) + f''(x-ct))\right) \\
&= c^2\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Now for the boundary conditions

$$\begin{aligned}
u(0,t) &= \frac{1}{2}(f(ct) + f(-ct)) \\
&= \frac{1}{2}\left(\sin\left(\frac{c\pi}{\ell}t\right) + \sin\left(-\frac{c\pi}{\ell}t\right)\right) \\
&= 0 \text{ since } \sin(-\theta) = -\sin\theta \\
&\quad \text{[This really only needs } f \text{ odd]} \\
u(\ell,t) &= \frac{1}{2}(f(\ell+ct) + f(\ell-ct)) \\
&= \frac{1}{2}\left(\sin\left(\pi+\frac{c\pi}{\ell}t\right) + \sin\left(\pi-\frac{c\pi}{\ell}t\right)\right) \\
&= \frac{1}{2}(-\sin\left(\frac{c\pi}{\ell}t\right) + \sin\left(\frac{c\pi}{\ell}t\right) \\
&\quad \text{using } \sin(\pi+\theta) = -\sin\theta \text{ and} \\
&\qquad \sin(\pi-\theta) = -\sin(-\theta) = \sin\theta \\
&= 0 \\
&\quad \text{[Can get by with } f \text{ odd and period } 2\ell]
\end{aligned}$$

and the initial data

$$\begin{aligned}
u(x,0) &= \frac{1}{2}(f(x) + f(x)) \\
&= f(x) \\
\frac{\partial u}{\partial t}(x,0) &= \frac{c}{2}(f'(x+ct) - f'(x-ct))\ |_{t=0} \\
&= \frac{c}{2}(f'(x) - f'(x)) = 0
\end{aligned}$$