Note: This material is contained in Kreyszig, Chapter 13.

Complex integration

We will define integrals of complex functions along curves in $\mathbb{C}$. (This is a bit similar to [real-valued] line integrals $\int_\gamma P\,dx + Q\,dy$ in $\mathbb{R}^2$.)

A **curve** is most conveniently defined by a parametrisation. So a curve is a function $\gamma\colon [a,b] \to \mathbb{C}$ (from a finite closed real intervale $[a,b]$ to the plane). We can imagine the point $\gamma(t)$ being traced out by a pen which is at position $\gamma(t)$ at time t. We can write $\gamma(t) = x(t) + iy(t)$ in terms of its real an imaginary parts.

Then we define $\gamma'(t) = x'(t) + iy'(t)$ (can be viewed as the tangent vector or velocity vector to the curve) and we will only be dealing with curves where $\gamma'(t)$ is defined and continuous.

A curve is called *closed* if $\gamma(a) = \gamma(b)$ (start and end point coincide).

A curve is called *simple* if it never goes though the same point twice (with the possible exception that $\gamma(a) = \gamma(b)$ is allowed — apart from this all $\gamma(t)$ have to be different points).

Example. A simple example to keep in mind is a circle, say the circle of radius $r > 0$ about the origin where we travel once around it anticlockwise starting and ending at the point r on the positive axis.

Then $\gamma_r\colon [0, 2\pi] \to \mathbb{C}$,

$$\gamma_r(t) = re^{it} = r\cos t + ir\sin t$$

is one obvious parametrisation.

Definition If $\gamma\colon [a,b] \to \mathbb{C}$ is a curve in $\mathbb{C}$ and $f(z)$ is a complex-valued function defined at least for all $z = \gamma(t)$, then we define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

(This last is a fairly ordinary integral, except that $f(\gamma(t))\gamma'(t)$ will have complex values. Say $f(\gamma(t))\gamma'(t) = p(t) + iq(t)$ (in terms of the real part $p(t)$ and imaginary part $q(t)$). Then the complex integral means simply

$$\int_a^b p(t) + iq(t)\,dt = \int_a^b p(t)\,dt + i\int_a^b q(t)\,dt$$

Technically we will require that these ordinary integrals of p and q should exist, but that will be ok in all our examples. Continuity of f and of $\gamma'(t)$ is enough to make the integral ok.

Example. Take γ_r as in the example above and $f(z) = 1/z$. Then we can explicitly compute

$$\int_{\gamma_r} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,dt = \int_0^{2\pi} i dt = 2\pi i$$

(In practice, we can rarely do the calculation so directly. Typically we will use theorems to simplify the curve first.)

Elementary properties (of complex integrals). The basic properties are reminiscent of those for line integrals in $\mathbb{R}^2$ (except that we now have complex values).

1. The exact parametrisation of the curve γ is not important, although the direction is. So for example, if we take the circle $|z| = r$ but parametrise it in a different way, while still going once around anticlockwise — say by $\sigma_r\colon [0,1] \to \mathbb{C}$ with $\sigma_r(t) = e^{2\pi i t}$, then the integral will not change. So
$$\int_{\gamma_r} f(z)\,dz = \int_{\sigma_r} f(z)\,dz$$
Changing the direction of the curve changes the the integral by a factor -1. For example in the case of the circle, $\mu_r\colon [0,2\pi] \to \mathbb{C}$ with $\mu_r(t) = e^{-2\pi i t}$ has $\int_{\mu_r} f(z)\,dz = -\int_{\gamma_r} f(z)\,dz$. These fact follow by ordinary substitution (or change of variables).

2. If $f(z) = F'(z)$ for some analytic $F(z)$ and $\gamma\colon [a,b] \to \mathbb{C}$ is a curve with all points $\gamma(t)$ in the set where $F(z)$ is analytic, then
$$\int_\gamma f(z)\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = [F(\gamma(t))]_{t=a}^b = F(\gamma(b)) - F(\gamma(a))$$
is the difference of the values $F(\text{end}) - F(\text{start})$.

3. In particular, if the integrand $f(z)$ has an analytic antiderivative $F(z)$ that works all along γ, then the exact path γ does not enter in to the value of $\int_\gamma f(z)\,dz$ (as long as γ stays in the set where F is analytic) and the integral will be 0 if the path is closed (start = end).

   If you look back at the last example you will see that the integral of $f(z) = 1/z$ around the closed curve γ_r was not zero. Thus there is no antiderivative of $1/z$ that works all the way around γ_r. [Recall that $\log z$ is an antiderivative of $1/z$ except on the negative axis. The jump we make in the argument $\arg(z)$ at the negative axis actually corresponds to the value $2\pi i$ of the integral.]

4. We can use Green's theorem for complex valued $P(x,y)$ and $Q(x,y)$. That is
$$\int_\gamma P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy$$
is true for complex-valued P, Q if γ is a simple closed curve in $\mathbb{R}^2$, R is the interior of γ and both P and Q are well-behaved inside and on γ.

   Here we interpret the integrals of complex things as the integral of the real part $+i$ times the integral of the imaginary part.

Theorem 1 (Cauchy's theorem) *If γ is a simple closed anticlockwise curve in the complex plane and $f(z)$ is analytic on some open set that includes all of the curve γ and all points inside γ, then*
$$\int_\gamma f(z)\,dz = 0$$

Proof. We write $dz = dx + i dy$ and use Green's theorem on

$$\int_\gamma f(z)\,dz = \int_\gamma f(z)\,dx + (if(z))\,dy = \iint_R \left(\frac{\partial(if(z))}{\partial x} - \frac{\partial f(z)}{\partial y} \right) dx\,dy$$

(with R denoting the interior of γ). If you recall the proof of the CR equations you will remember that (because the limit defining $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ can be taken in any direction or in all directions at once

$$f'(z) = \frac{\partial f(z)}{\partial x} = \frac{1}{i}\frac{\partial f(z)}{\partial y}$$

It follows that the integrand of the double integral we got from Green's theorem

$$\frac{\partial(if(z))}{\partial x} - \frac{\partial f(z)}{\partial y} = i\left(\frac{\partial f(z)}{\partial x} - \frac{1}{i}\frac{\partial f(z)}{\partial y}\right) = 0$$

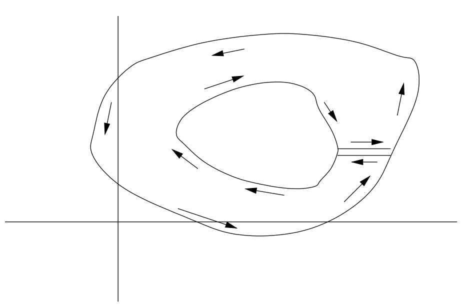
and so we get $\int_\gamma f(z)\,dz = 0$. [Another way to do this is to write $f(z) = u + iv$ and use the CR equations to get the integrand of the double integral to be zero.]

Remark. It is vital that there are no bad points of $f(z)$ inside or on γ. Look again at the last example and see that $f(z) = 1/z$ is fine everywhere except at $z = 0$. This can (and does in that case) make the integral nonzero.

Corollary 2 *Suppose we have two anticlockwise simple closed curves γ_1 and γ_2 with one entirely contained in the interior of the other. Suppose $f(z)$ is analytic on some open set that includes both γ_1 and γ_2 and the region between the two curves. Then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

(The proof involves making a 'narrow bridge' between the two curves and a simple closed curve Γ that goes almost once around the outer curve, in across one side of the bridge, the wrong way around the inner curve and back across the bridge.



$f(z)$ will be analytic on and inside Γ and then $\int_\Gamma f(z)\,dz = 0$ by Cauchy's theorem. Let the width of the bridge tend to zero and we find that we get the result we want because the integral along the bridge in different directions cancel.)

Example. Looking back at the example $\int_{\gamma_r} 1/z\,dz$ we saw they all turned out to be $2\pi i$ no matter what the radius $r > 0$ was. We can now see that this independence of r follows because of (the Corollary to) Cauchy's theorem. Also we can see that we will also get the same $2\pi i$

for integrals around more complicated curves that go once anticlockwise around the origin. For example the ellipse $\sigma\colon [0, 2\pi] \to \mathbb{C}$ with

$$\sigma(t) = 4\cos t + 3i\sin t$$

contains γ_r if $r < 3$. So $\int_\sigma 1/z\,dz = \int_{\gamma_2} 1/z\,dz = 2\pi i$.

Theorem 3 (Cauchy's integral formula) *Suppose that γ is a simple closed anticlockwise curve in the complex plane and $f(z)$ is analytic on some open set that includes all of the curve γ and all points inside γ. Then for any point z_0 inside γ we have*

$$\int_\gamma \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$$

(We will not proved this but the idea is that the integral will remain the same if we replace γ by a small circle around z_0. If we let the radius of the small circle $\to 0$ then we can show that the integral must be very close to $2\pi i f(z_0)$. As the integral is idependent of the radius, it must actually be $2\pi i f(z_0)$.)

Example. Consider

$$\int_{\gamma_2} \frac{z^2 + z + 1}{z^2 + 2z - 3}\,dz$$

The integrand is bad at two points because the denominator is zero at two places.

$$\frac{z^2 + z + 1}{z^2 + 2z - 3} = \frac{z^2 + z + 1}{(z + 3)(z - 1)}$$

(The bad points are $z = 1$ and $z = -3$.) Only one of these bad points $z = 1$ is inside γ_2. We can in fact write the integral as

$$\int_{\gamma_2} \frac{z^2 + z + 1}{z^2 + 2z - 3}\,dz = \int_{\gamma_2} \frac{f(z)}{z - 1}\,dz \text{ where } f(z) = \frac{z^2 + z + 1}{z + 3}$$

and then the Cauchy integral formula gives the answer $2\pi i f(1) = 2\pi i \frac{3}{4} = 3\pi i/2$ for the integral we started with.

It did not matter that the curve was exactly the circle of radius 2, only that it went around 1 but not around -3.

Power series. One can make use of Cauchy's integral formula to prove that every analytic function $f(z)$ can be represented by a power series in any disc where it is analytic.

If $f(z)$ is analytic in an open set that includes the disc $\{z \in \mathbb{C} : |z - z_0| < r\}$ of radius $r > 0$ about z_0, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } z \text{ with } |z - z_0| < r.$$

The coefficients a_n can be represented as integrals

$$a_n = \frac{1}{2\pi i} \int_{|z - z_0| = s} \frac{f(z)}{(z - z_0)^{n+1}}\,dz \text{ for any } 0 < s < r, \text{ or as } a_n = \frac{f^{(n)}(z_0)}{n!}$$

Example. Take $f(z) = 1/z$ and $z_0 = 2$. Then $f(z)$ is analytic for $|z - z_0| < 2$ and so there is a power series for $f(z)$ there.

A more complicated example is $f(z) = \dfrac{e^z}{(z-1)(z-2)}$. For any z_0 different from 1 and 2, there is power series for $f(z)$ in the largest disc $|z - z_0| < r$ that misses 1 and 2. Specifically r is the shorter of the two distances $|z - 1|$ and $|z - 2|$.