Note: This material is contained in Kreyszig, sections 12.1–12.5, 14.1–14.2.

Rest of recap

Analytic is defined via complex differentiable. $w = f(z)$ complex differentiable at z_0 means the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (as a complex number).

To be able to take such a limit at all one needs $f(z)$ to make sense (or in other words be defined) for all z within a certain positive distance from z_0.

Another way to put that: z_0 must be an interior point of the domain of definition of f. We will normally assume that that every point of the domain of f is an interior point of that domain. The technical term for this is *open set*. An open set in the complex plane $\mathbb{C}$ is one where each of its points in an interior point. In graphical terms, an open set is one that includes none of its 'edge' or 'boundary' points. The whole complex plane has no edge and so is open.

An function $w = f(z)$ defined for z in an open set G is called *analytic on* G if $f'(z_0)$ exists for each $z_0 \in G$.

Note: familiar differentiation formulae like the product rule and the Chain rule and $\frac{d}{dz} z^n = nz^{n-1}$ work for complex analytic functions.

Cauchy-Riemann equations (a system of 2 PDEs)

$f(z) = u(z) + iv(z)$ with $u(z), v(z)$ real-valued. Or $f(x + iy) = u(x + iy) + iv(x + iy)$. Treat $u(x + iy)$ as a function $u(x, y)$ of 2 real variables. Same for $v(x, y)$. The CR equations say

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

A function $w = f(z)$ is analytic if and only if the corresponding u, v satisfy the CR-equations.

Consequence of CR equations: u, v satisfy Laplace's equation.

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \end{aligned}$$

Example $f(z) = z^3$. We can expand

$$f(x + iy) = (x + iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

and so in this case $u(x, y) = x^3 - 3xy^2$, $v(x, y) = 3x^2y - y^3$.

We could check that the CR equations hold here. For example $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ and $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$.

From the CR equations, if you know u you can find v. (Well not quite. You can add a constant to v.) That works in reverse. If you know v you can find u (up to constants). [1]

This is an example of the rigidity (or predictability) of analytic functions. That rigidity is what makes them special and is the underlying reason why many things work nicely.

Power series

Many analytic functions arise as sums of power series.

As in the real case a power series has a center which we will denote z_0 and the series is a sum of powers $(z - z_0)^n$ with coefficients in front. So a power series centered at $z_0 \in \mathbb{C}$ is a series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

An infinite sum like this needs a definition. We will not go into it in any great detail but the infinite sum is defined to be the limit of the (finite) partial sums

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (z - z_0)^k.$$

Example. Take the case where each $a_n = 1$ and then the power series

$$\sum_{n=0}^{\infty} (z - z_0)^n = 1 + (z - z_0) + (z - z_0)^2 + \cdots$$

is a geometric series. We can work out the partial sums explicitly using the formula $a + ar + ar^2 + \cdots + ar^{n-1} = a(1 - r^n)/(1 - r)$ (valid for $r \neq 1$). We get

$$s_n = \sum_{k=0}^{n} (z - z_0)^k = \frac{1 - (z - z_0)^{n+1}}{1 - (z - z_0)} \qquad \text{if } z - z_0 \neq 1 \text{ and } = n + 1 \text{ if } z = z_0 + 1$$

For $\lim_{n\to\infty} s_n$ to exist we need $|z - z_0| < 1$ and then the limit is $1/(1 - (z - z_0))$.

Radius of convergence. This fact that the values of z where the power series converges is a disc with center at the center of the power series is always the case, not just for this example.

For every power series there is a radius $R \geq 0$ (but we can have $R = \infty$ sometimes) so that the series converges inside the disc where $|z - z_0| < R$ and does not converge at all for any z with $|z - z_0| > R$.

[1] If the domain is made up of several unrelated parts (technically called disconnected; an example is the union of two open and non-intersecting discs) then you can use different constants on the different parts. Usually we don't have to deal with this case.

If the domain has a complicated shape we can sometimes have trouble choosing the constant in a consistent way.