Stokes's theorem

Suppose S is an oriented suface in $\mathbb{R}^3$ (means that there is a continuously varying unit normal at all points of S) which is well behaved and bounded by a closed curve C (or possibly a finite number of closed curves) and suppose C is oriented so that S is to the left. Let $\mathbf{F} = \mathbf{F}(x, y, z) = [F_1, F_2, F_3]$ be a vector field well-behaved on S and its boundary C. Then

$$\iint_S (\nabla \times \mathbf{F}).\mathbf{n}\, dA = \oint_C \mathbf{F}.d\mathbf{x}$$

Examples. (i) Green's theorem (S planar). (ii) S a closed surface like a sphere (no C).

Green's theorem

Another way to write the theorem (equivalant way) is: Assume $\mathbf{F}(x, y) = [F_1(x, y), F_2(x, y)]$ is well behaved in a region of the plane that includes an anticlockwise simple closed curve C and its interior R. Suppose $\mathbf{n}$ is the unit normal to C pointing outwards. Then

$$\oint_C \mathbf{F}.\mathbf{n}\, ds = \iint_R \nabla.\mathbf{F}\, dx\, dy$$

($ds =$ arclength).

Reason. $\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$ is the unit tangent vector to C. $\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$. Hence $\mathbf{F}.\mathbf{n}\, ds = F_1\, dy - F_2\, dx$.

Heat equation

We consider a heated solid object. $u(x, y, z, t) =$ temperature at position (x, y, z) and at time t.

Law of heat flow: Heat will flow in direction of maximum decrease of termperature, that is in the direction of $-\nabla u = -[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}]$ at a rate proportional to $\|\nabla u\|$. Flow rate $= K\|\nabla u\|$ with $K =$ thermal conductivity.

We assume $K > 0$ constant. Consider any subregion

R inside the solid with boundary surface S.

$\iint_S -K(\nabla u).\mathbf{n}\, dS =$ rate of heat flow out of S.
(Pictorially $(\nabla u).\mathbf{n}\, dS = \|\nabla u\| \cos\theta\, dS = \|\nabla u\|$ area of infinitesimal section dS times cosine of angle with direction of flow (effective cross sectional area for flow from dS).

On the other hand, total heat inside R at any time is $\iiint_R \sigma\rho u\, dx\, dy\, dz$ (with $\sigma =$ specific heat, $\rho =$ density of material). Computing rate of decrease of heat (or rate of heat loss) in two ways

$$\iint_S -K(\nabla u).\mathbf{n}\, dS = -\frac{\partial}{\partial t}\iiint_R \sigma\rho u\, dx\, dy\, dz$$

Apply Gauss' theorem on the left and bring the deriviative inside the integral on the right

$$\iiint_R -K\,\mathrm{div}\,(\nabla u)\, dx\, dy\, dz \;=\; \iiint_R -\sigma\rho\frac{\partial u}{\partial t}\, dx\, dy\, dz$$

$$\iiint_R (\sigma\rho\frac{\partial u}{\partial t} - K\nabla^2 u)\, dx\, dy\, dz \;=\; 0$$

True for all small regions R inside solid. Forces

$$\frac{\partial u}{\partial t} = \frac{K}{\sigma\rho}\nabla^2 u$$

or

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$$

with $c^2 = K/(\sigma\rho) =$ thermal diffusivity of the material.

Remark. If the heat distribution is in a steady state (*i.e.* no t dependence in u) then $\partial u/\partial t = 0$. Hence $\nabla^2 u = 0$ (u satisfies Laplace's equation).

Remark. To derive the Heat equation for 2-dimensions (no z) can either use 2-D version of the divergence theorem (= version of Green's theorem stated after Stokes' above) or just assume $u(x, y, z, t)$ independent of z.

Remark. For 1-dimensional heat equation, can either use latter aproach (assume no y or z dependence) or look at thin rod with temperature $u(x,t)$ and heat flow proprtional to $-\partial u/\partial x$. Instead of Gauss' theorem use

$$\int_{\alpha}^{\beta} \frac{\partial^2 u}{\partial x^2}\, dx = \frac{\partial u}{\partial x}(\beta, t) - \frac{\partial u}{\partial x}(\alpha, t)$$

Exercise: Try to work this argument through.