Recap on vector calculus

div, grad $= \nabla$ and curl

For a scalar function $f(x, y, z)$

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

(vector function of (x, y, z)).

Also have a 2 dimensional version.

Operator notation

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

Directional derivative of f in direction (of a unit vector) $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = [v_1, v_2, v_3]$ is

$$D_{\mathbf{v}}f = (\nabla f).\mathbf{v} = \frac{\partial f}{\partial x}v_1 + \frac{\partial f}{\partial y}v_2 + \frac{\partial f}{\partial z}v_3$$

Recall that $D_{\mathbf{v}}f$ is largest (at a given (x, y, z)) when v is in the direction of ∇f.

For a vector function $\mathbf{F}(x,y,z) = [F_1, F_2, F_3] = [F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

$$\text{div}\,\mathbf{F} = \nabla.\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

(which is a scalar quantity).

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

Some formulae

$$\text{div}\,(\text{grad}\, f) = \nabla.(\nabla f)$$

$$= (\nabla.\nabla)f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

($= \triangle f =$ Laplacian of f)

$$\text{curl}\,\text{grad}\, f = \nabla \times (\nabla f) = 0$$

For a 2-dimensional vector field (function) $\mathbf{F}(x, y) = [F_1(x, y), F_2(x, y)]$, can view it as a 3-dimensional field $[F_1, F_2, 0]$ and then

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left[0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

Greens Theorem

For $\mathbf{F}(x,y) = [F_1(x,y), F_2(x,y)]$ well behaved in a region of the plane that includes an anticlockwise simple closed curve C and its interior R

$$\oint_C F_1 dx + F_2 dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy$$

or, in terms of the upward normal vector $\mathbf{k}$ to the plane

$$\oint_C \mathbf{F}.d\mathbf{x} = \iint_R (\nabla \times \mathbf{F}).\mathbf{k}\, dx\, dy$$

More generally R can be a bounded region with several holes and C the boundary of R (possibly several parts to C) but C must be traversed so as to keep R on the left.

If C is parametrised by $\mathbf{x} = \mathbf{x}(t)$ for $a \leq t \leq b$, then

$$\oint_C \mathbf{F}.d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)).\mathbf{x}'(t)\,dt$$

Divergence (Gauss') Theorem

R now a bounded region (volume) in space, S its boundary surface, $\mathbf{n} = \mathbf{n}(x, y, z) =$ the outward unit normal to S at a point (x, y, z) of S and $\mathbf{F}$ well-behaved on both R and S. Then

$$\iint_S (\mathbf{F}.\mathbf{n})\, dA = \iiint_R \nabla.\mathbf{F}\, dx\, dy\, dz$$

Note: If we write S parametrically as $\mathbf{r} = \mathbf{r}(u, v)$, then a normal vector is $\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$, unit normal $\mathbf{n} = \pm\mathbf{N}/|\mathbf{N}|$. $dA = |\mathbf{N}|\, du\, dv$. So $(\mathbf{F}.\mathbf{n})\, dA = \pm(\mathbf{F}.\mathbf{N})\, du\, dv$