UNIVERSITY OF DUBLIN

XMA3211?

TRINITY COLLEGE

Faculty of Science

school of mathematics

JS & SS Mathematics **Trinity Term 2006**

Course 321 — Modern Analysis

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Credit will be given for the best 6 questions answered. Logarithmic tables are available if required.

1. (a) State the axiom of choice and give a definition of any terminology involved in the statement.

   (b) Let $f: X \to Y$ be a surjective function. Show that there exists a mapping $g: Y \to X$ such that $f \circ g$ is the identity map on Y.

   Show that this is in fact equivalent to the Axiom of Choice. [Hint: Given a family of sets $\{A_i: i \in I\}$, consider the family of disjoint sets given by $B_i = A_i \times \{i\}$. Let $f: \bigcup_i B_i \to I$ be the function which has the value i on B_i.]

   (c) Show that there exists a discontinuous function $f: \mathbb{R} \to \mathbb{R}$ which satisfies the identity $f(x+y) = f(x) + f(y)$. [Hint: Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and use the fact that $\mathbb{R}$ has a basis over $\mathbb{Q}$ that contains 1. Take $f(1) = 0$ and $f(x) = 0$ on other basis elements.]

2. (a) Define the terms *partial order*, *linear order*, *well order* and *ordinal number*.

   (b) Explain why there is a cardinality associated with every ordinal number and give a brief explanation of why there is an ordinal with any given cardinality.

   (c) State Zorn's lemma and explain the terminology involved in the statement.

   (d) Show that in any inner product space, there exist maximal orthonormal subsets.

3. (a) Define the terms *first countable*, *second countable* and *separable* for topological spaces. Show that every second countable topological space is first countable. Show that every second countable topological space is separable.

   (b) Show that in a metric space (X, d) the collection $\mathcal{B}_x = \{B(x, 1/n) : n \in \mathbb{N}\}$ of all open balls of radius $1/n$ centered at a point $x \in X$ forms a neighbourhood base at x (in the metric topology).

   (c) Let X be an infinite set with the cofinite topology. (The open subsets are the empty set and subsets with finite complements in X.) Show that a sequence in X with distinct terms has every point of X as a limit.

   In case X is uncountable show that X is not first countable.

4. (a) Define the terms *net* and *subnet*, and explain the associated terminology.

   Show that a net $(x_\lambda)_{\lambda\in\Lambda}$ in a topological space X has a limit point $x \in X$ if and only if $(x_\lambda)_{\lambda\in\Lambda}$ has a subnet which converges to x.

   (b) Define *compactness* for topological spaces. Show that a topological space $(X, \mathcal{T})$ is compact if and only if each family $\mathcal{F}$ of closed subsets of X with the finite intersection property has empty intersection.

   (c) Show that a topological space is compact if and only if every net in the space has a limit point.

5. (a) Define *normality* for topological spaces and prove that compact Hausdorff spaces are normal.

   (b) If X is a compact Hausdorff space and $x_0, x_1 \in X$ are two different points of X, show that there is a continuous $f\colon X \to [0,1]$ with $f(x_0) = 0$ and $f(x_1) = 1$.

   (c) Define the notion of *compactification* of a topological space X and define what is meant by a *Tychonoff space*.

   (d) Outline a proof that every Tychonoff space has a Stone-Čech compactification.

6. (a) Define *boundedness* for a linear transformation between normed spaces and show that it is equivalent to continuity and to uniform continuity of the transformation. Define the *operator norm* of a bounded linear operator.

   (b) Define the Banach spaces usually denoted $L^p([0,1])$ $(1 \le p < \infty)$ and $C([0,1])$.

   (c) Show that the inclusion map $C([0,1]) \to L^p([0,1])$ is continuous, linear, has operator norm 1, but is not surjective.

7. (a) Define what is meant by a *Hilbert space*.

   (b) State and prove Bessel's inequality.

   (c) Outline a proof that every separable infinite dimensional Hilbert space is isometrically isomorphic to ℓ^2.

8. (a) Prove that the 'standard basis' of the Hilbert space ℓ^2 is an orthonormal basis for ℓ^2'.

   (b) Show that the sequence space c_0 cannot be a Hilbert space in the usual supremum norm on c_0. [Hint: parallelogram identity.]

   (c) Show that the sequence space c_0 cannot be a Hilbert space in any norm equivalent to the usual supremum norm. [Hint: Is it reflexive?]

   (d) Show that there is $f \in L^2([0, 2\pi])$ with

$$\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{|n|+1} \quad (\forall n \in \mathbb{Z}).$$

9. (a) If E is a normed space and F is a Banach space, show that

$$\mathcal{L}(E, F) = \{T : E \to F : T \text{ a bounded linear operator}\}$$

   is a Banach space in the operator norm (when we define vector space operations on $\mathcal{L}(E, F)$ by $(T + S)(x) = T(x) + S(x)$, $(\lambda T)(x) = \lambda(T(x))$).

   (b) Let X be a nonempty compact Hausdorff space and $f : X \to X$ continuous. Show that there exists a nonempty closed subset $A \subset X$ with $f(A) = A$. [Hint: Put

$$A = X \cap f(X) \cap f(f(X)) \cap \cdots$$

   To show $A \subset f(A)$, fix $a \in A$ and consider $f^{-1}(a) \cap A = \left(f^{-1}(a) \cap X\right) \cap \left(f^{-1}(a) \cap f(X)\right) \cap \cdots$.]