Mathematics 321 2008–09
Exercises 8
[Due Monday March 30th.]

1. Suppose Z is a Banach space and $X, Y \subset Z$ are two closed subspaces with $X \cap Y = \{0\}$ and $X + Y = Z$ (that is $\{x + y : x \in X, y \in Y\} = Z$).

   Show that the map $P\colon Z \to Z$ given by $P(x + y) = x$ is

   (a) well-defined and linear;

   (b) bounded;

   (c) idempotent (that is satisfies $P \circ P = P$); and

   (d) has $\ker P = Y$.

2. If Z is a normed space and $P\colon Z \to Z$ is a bounded linear idempotent map [we call P a (bounded) 'projection' in that case], show that Z is isomorphic to $P(Z) \oplus_1 \ker P$.

   [Hint: Show that $z - P(z) \in \ker P$ for $z \in Z$. Write $z \in Z$ as $z = P(z) + (z - P(z))$. Show that the map $S\colon Z \to P(Z) \oplus_1 \ker P$ given by $S(z) = (P(z), z - P(z))$ is bounded, linear and inverse to the map $T\colon P(Z) \oplus_1 \ker P$ given by $T(x, y) = x + y$.]

3. If $X \subset Z$ is a one-dimensional subspace of a normed space Z, show that there is a bounded projection of Z onto X (*i.e.* with range X). [Hint: Hahn-Banach]

4. If Z is a Banach space and $X \subset Z$ is a closed subspace of codimension one (*i.e.* the quotient Z/X has dimension one), show that there is a bounded projection of Z onto X. [Hint: Recall that the quotient vector space Z/X is the set of cosets $z + X$ with $z \in Z$ and we have $z_1 + X = z_2 + X \iff z_1 - z_2 \in X$. Choose a vector $y \in Z \setminus X$. Every $z \in Z$ can be expresse as $z = x + \lambda y$ for $\lambda \in \mathbb{K}$. Look at $Y = \{\lambda y : \lambda \in \mathbb{K}\}$.]

5. If H is a Hilbert space and $S \subset H$ is an orthonormal subset, show that there is an orthonormal basis for H that contains S.

6. If H is a Hilbert space and $M \subset H$ is a closed linear subspace, let $M^\perp = \{y \in H : \langle x, y\rangle = 0 \forall x \in M\}$. Show that each element $z \in H$ can be expressed as $z = x + y$ for $x \in M$ and $y \in M^\perp$. [Hint: Extend an orthonormal basis of M to get one for H.]

7. If H is a Hilbert space and $M \subset H$ is a closed linear subspace, show that there is an isometric linear isomorphism $: M \oplus_2 M^\perp \to H$ given by $(x, y) \mapsto x + y$.

8. If H is a Hilbert space and $M \subset H$ is a closed linear subspace, show that the projection $P\colon H \to H$ with $P(H) = M$ and $\ker P = H^\perp$ is self-adjoint (that is, satisfies $P^* = P$).