Mathematics 321 2008–09
Exercises 7
[Due Thursday March 5th.]

1. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. For $1 \leq p \leq \infty$, verify that we can define a norm on $E \oplus F$ by
$$\|(x,y)\|_p = \|(\|x\|_E, \|y\|_F)\|_p = (\|x\|_E^p + \|y\|_F^p)^{1/p}$$
(in terms of the p-norm of $\mathbb{R}^2$).
(We denote the normed space $(E \oplus F, \|\cdot\|_p)$ by $E \oplus_p F$.)

2. With E and F as before and $1 \leq p, r \leq \infty$, show that the 'identity' map $T\colon E \oplus_p F \to E \oplus_r F$, given by $T(x,y) = (x,y)$, is an isomorphism (of normed spaces — bounded linear isomorphism with a bounded inverse).

3. With E and F as before and $1 \leq p \leq \infty$, let $\tilde{E} = \{(x,0) : x \in E\} \subset E \oplus_p F$ and $\tilde{F} = \{(0,y) : y \in F\} \subset E \oplus_p F$.
Show that $\tilde{E} \cap \tilde{F} = \{0\}$ and that each element $z \in E \oplus_p F$ can be expressed uniquely in the form $z = x + y$ with $x \in \tilde{E}$, $y \in \tilde{F}$.

4. With E and F as before and $1 \leq p \leq \infty$, show that the dual space of $E \oplus_p F$ can be identified with $E^* \oplus_q F^*$ (where $1/p + 1/q = 1$).
More precisely, show that the map $T\colon E^* \oplus_q F^* \to (E \oplus_p F)^*$ given by $T(\phi,\psi)(x,y) = \phi(x) + \psi(y)$ (for $\phi \in E^*$, $\psi \in F^*$, $x \in E$, $y \in F$) is an isometric isomorphism.
[Hint: Use Hölder's inequality to show $\|T(\phi,\psi)\| \leq \|(\phi,\psi)\|_q$. To show the reverse inequality, fix (ϕ,ψ) and $\varepsilon > 0$. There are unit vectors $x \in E$, $y \in F$ so that $|\phi(x)| > \|\phi\| - \varepsilon$ and $|\psi(y)| > \|\psi\| - \varepsilon$. Show you can assume $\phi(x) = |\phi(x)|$ and $\psi(y) = |\psi(y)|$ (use rotations). Take $z = (tx, sy)$ for suitable $0 \leq s,t \leq 1$, $s^p + t^p = 1$ and consider $T(\phi,\psi)(z)$.
You still have to show T is surjective.]

5. Suppose now Z is a normed space and $X, Y \subset Z$ are two closed subspaces with $X \cap Y = \{0\}$ and $X + Y = Z$ (that is $\{x + y : x \in X, y \in Y\} = Z$).
   (a) Show that each $z \in Z$ can be uniquely expressed in the form $z = x + y$ for $x \in X$, $y \in Y$.
   (b) Show that the map $T\colon X \oplus_1 Y \to Z$ given by $T(x,y) = x + y$ is linear, bounded and bijective.
   (c) If Z is complete, show that T has a bounded inverse. [So it is an isomorphism of Banach spaces. Hint: Open mapping theorem.]

6. If H and K are Hilbert spaces, show that there is an inner product on $H \oplus_2 K$ that gives rise to the norm.