Mathematics 321 2008–09
Exercises 6
[Due Friday February 13th.]

1. If $(S, \leq)$ is a partially ordered set, show that the union of any chain of linearly ordered subsets of S is again a linearly ordered subset of S.

2. Let V be the vector space of all sequences $(x_1, x_2, \ldots, x_n, \ldots)$ of real numbers which become zero after some term x_n (n depends on the sequence). We refer to these as finitely nonzero sequences of real numbers. V is a vector space over $\mathbb{R}$ when we define

$$\begin{aligned}(x_1, x_2, \ldots, x_n, \ldots) + (y_1, y_2, \ldots, y_n, \ldots) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots)\\ \lambda(x_1, x_2, \ldots, x_n, \ldots) &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots)\end{aligned}$$

   Exhibit a basis for V (with proof).

3. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Let $E \subset \mathbb{R}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Prove that E is uncountable.

4. Let V be a vector space over a field F and suppose $S \subset V$ spans V. If $A \subset S$ is a linearly independent subset, show that there exists a basis B of V with $A \subset B \subset S$.

5. Let V be a vector space over a field F and $v_0 \in V$ a non-zero vector in V.

   (a) If V does not have dimension one, show that V has a basis which does not contain v_0.

   (b) Is there a counterexample in the case when V has dimension one?