Mathematics 321 2008–09
Exercises 5
[Due Friday January 30th.]

1. Let E and F be Banach spaces and $T\colon E \to F$ a linear transformation (not assumed to be bounded). Let

$$S_T = \{y \in F : \exists \text{ a sequence } (x_n)_{n=1}^\infty \text{ in } E \text{ with } \lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} T(x_n) = y\}$$

   (a) If T is bounded, show that $S_T = \{0\}$.

   (b) If $S_T = \{0\}$ show that the graph of T is closed.

   (c) If $S_T = \{0\}$ show that T is bounded.

   (S_T is known as the 'separating space' of T.)

2. Let $f\colon X \to Y$ be a surjective function. Show that there exists a mapping $g\colon Y \to X$ such that $f \circ g$ is the identity map on Y.

   Show that this is in fact equivalent to the Axiom of Choice. [Hint: Given a family of sets $\{A_i\colon i \in I\}$, consider the family of disjoint sets given by $B_i = A_i \times \{i\}$. Let $f\colon \bigcup_i B_i \to I$ be the function which has the value i on B_i.]

3. Draw the picture of $(\mathcal{P}(X), \subseteq)$ for $X = \{0, 1, 2\}$.

   Give an example of a chain in $(\mathcal{P}(X), \subseteq)$ in this case.

4. Give an example of a chain in a partially ordered set such that the chain has no upper bound.

5. Give an example of a subset of a partially ordered set with a maximal element but no upper bound.

6. Let $S =$ the set of all proper subsets of $\mathbb{N}$ which are nonempty ($\neq \emptyset$ and $\neq \mathbb{N}$), and order S by set inclusion.

   Find a maximal element of S and a minimal element. (How should a minimal element be defined?)

   Is the maximal element you found the largest element?

7. Show that every group has a maximal abelian subgroup. [Note not necessarily a proper subgroup.]