Mathematics 321 2008–09
Exercises 4
[Due Monday January 12th.]

1. Let E be a normed space and define $B_E = \{x \in E : \|x\| < 1\} = B(0,1)$. [We refer to B_E as the 'unit ball' of E.]

   (i) For $r > 0$ show that $B(0,r) = rB_E$.
   [Here rB_E means $\{rx : x \in B_E\}$.]

   (ii) For $r > 0$ and $x_0 \in E$, show that $B(x_0, r) = x_0 + B(0,r) = x_0 + rB_E$.
   [Here the notation $x_0 + B(0,r)$ means $\{x_0 + x : x \in B(0,r)\}$.]

2. Consider a finite closed interval $[a,b] \subset \mathbb{R}$ and let μ denote normalised length measure on $[a,b]$. So that means $d\mu(x) = \frac{1}{b-a}\,dx$ and for (measurable and integrable) functions $f\colon [a,b] \to \mathbb{R}$, $\int_{[a,b]} f(x)\,d\mu(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

   Let $1 \leq p < \infty$, and consider the two L^p spaces $L^p([a,b])$ (where we will use $\|f\|_p$ for the norm) and $L^p([a,b],\Sigma,\mu)$. Here Σ means the measurable subsets of $[a,b]$, and we will denote the norm of the 'abstract' L^p space $L^p([a,b],\Sigma,\mu)$ by $\|f\|_{L^p(\mu)}$ so as to distingusih it from $\|f\|_p$.

   (i) Show that as sets, $\mathcal{L}^p([a,b])$ and $\mathcal{L}^p([a,b],\Sigma,\mu)$ coincide.

   (ii) Show that as sets, $L^p([a,b])$ and $L^p([a,b],\Sigma,\mu)$ coincide.
   [That means that the equivalence relation we use to pass from $\mathcal{L}^p$ to L^p is the same in both situations.]

   (iii) Express $\|f\|_p$ in terms of $\|f\|_{L^p(\mu)}$.

   (iv) If $1 \leq p_1 < p_2 < \infty$, how are $L^{p_1}([a,b],\Sigma,\mu)$ and $L^{p_2}([a,b],\Sigma,\mu)$ related? And how are the norms $\|f\|_{L^{p_1}(\mu)}$ and $\|f\|_{L^{p_2}(\mu)}$ related?

   (v) Again if $1 \leq p_1 < p_2 < \infty$, how are $L^{p_1}([a,b])$ and $L^{p_2}([a,b])$ related? And how are the norms $\|f\|_{p_1}$ and $\|f\|_{p_2}$ related?

3. Let X and Y be normed spaces (over $\mathbb{K}$) and denote by $B(X,Y)$ the set of bounded linear operators from X to Y.

   Let $T, S \in B(X,Y)$ and $\lambda \in \mathbb{K}$.

   (i) Define $T + S\colon X \to Y$ by the rule $(T+S)(x) = T(x) + S(x)$. Show that $T+S$ is linear, that $T+S$ is bounded and that $\|T+S\| \leq \|T\| + \|S\|$.
   [Here $\|T\|$ means the operator norm of T, and similarly for $\|S\|$ and $\|T+S\|$.]

   (ii) Define $\lambda T\colon X \to Y$ by the rule $(\lambda T)(x) = \lambda T(x)$. Show that λT is linear, that λT is bounded and that $\|\lambda T\| = |\lambda|\|T\|$.

(iii) Show that $B(X,Y)$ is a vector space when we use the notions of addition and multiplication by scalar just defined.

(iv) Show that $B(X,Y)$ becomes a normed space with the vector space structure in the previous part of the question and the operator norm.

(v) Supose $(T_n)_{n=1}^{\infty}$ is a Cauchy sequence in $B(X,Y)$ (with the operator norm). Show that for each $x \in X$, $(T_n x)_{n=1}^{\infty}$ is a Cauchy sequence in Y.

[You could read this as follows: A Cauchy sequence of operators (in the operator norm) is pointwise Cauchy.]

(vi) Assuming that Y is a Banach space (complete as well as normed) show that $B(X,Y)$ is a Banach space (in the operator norm).

[Hint: The previous part should help. If $(T_n)_{n=1}^{\infty}$ is a Cauchy sequence in $B(X,Y)$ we can define a function $T\colon X \to Y$ by $Tx = \lim_{n\to\infty} T_n x$. (Why?) You need to check that $T \in B(X,Y)$ and then that $\lim_{n\to\infty} T_n = T$ in $(B(x,Y), \|\cdot\|_{op})$.]