Mathematics 321 2008–09
Exercises 3
[Due Friday November 28th.]

1. If $(X, \|\cdot\|)$ is a normed space and $a \in X$, show that the translation map $x \mapsto x + a$ on X is continuous.

2. If $(X, \|\cdot\|)$ is a normed space (over $\mathbb{K}$) and $\lambda \in \mathbb{K}$, show that the dilation map $x \mapsto \lambda x$ on X is continuous.

3. Show that $\ell^p \subset c_0$ for $1 \leq p < \infty$.

4. Show that c_0 is a separable metric space.

   [Hint: Recall that a metric space (X, d) is called separable if there is a countable dense subset $S \subset X$. Countable means S is finite or else the elements of S can be listed in an infinite sequence $S = \{s_1, s_2, \ldots\}$ of elements. To say S is dense in X means that the closure $\bar{S} = X$.

   In $\mathbb{R}$ the rationals $\mathbb{Q}$ are a countable dense subset. In $\mathbb{C}$, $\{q_1 + iq_2 : q_1, q_2 \in \mathbb{Q}\}$ (the complex numbers with real and imaginary parts both rational) forms a countable dense subset.

   In c_0, consider the elements of the form $x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ (finitely nonzero sequences) with all terms x_j rational (or complex rational if $\mathbb{K} = \mathbb{C}$).]

5. Show that ℓ^p is a separable metric space for each $1 \leq p < \infty$.

6. Show that ℓ^∞ is not separable.

   [Hint: Metric subspaces of separable metric spaces are known to be separable. In ℓ^∞ the subset consisting of sequences $x = (\epsilon_1, \epsilon_2, \ldots)$ where $\epsilon_j = 0$ or 1 for each j. is uncountable and discrete.]