Mathematics 321 2008–09
Exercises 2
[Due Monday November 10th.]

1. If $(X, \|\cdot\|)$ is a normed space and $x \in X$, $r > 0$, show that $\bar{B}(x, r)$ is the closure of the open ball $B(x, r)$.

2. Suppose (X, d) is a metric space, $S \subset X$ is a subset and we give S the induced (or subspace) metric d_S given by $d_S(s_1, s_2) = d(s_1, s_2)$ for $s_1, s_2 \in S$.

   If (S, d_S) is a complete metric space, show that $S \subset X$ must be closed. On the other hand, if (X, d) is complete and S is closed in X, show that (S, d_S) is complete.

3. Suppose (X, d) is a metric space, and suppose $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X. If there is a subsequence $(x_{n_j})_{j=1}^\infty$ which converges in (X, d) to a limit $y \in X$, show that $\lim_{n\to\infty} x_n = y$.

   (In other words if a Cauchy sequence has a convergent subsequence, then the whole sequence must converge to the same limit.)

4. If (X, d_X) and (Y, d_Y) are metric spaces and $f\colon X \to Y$ is a function, then f is called uniformly continuous if it satisfies:

   Given $\varepsilon > 0$ there exists $\delta > 0$ so that
   $$x_1, x_2 \in X, d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$$

   (a) Show that uniformly continuous maps (from one metric space to another) are continuous.

   (b) If $f\colon X \to Y$ is uniformly continuous and $(x_n)_{n=1}^\infty$ is a Cauchy sequence in (X, d_X) show that $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence in (Y, d_Y).

5. Give an example of two metric spaces (X, d_X) and (Y, d_Y), a continuous map $f\colon X \to Y$ and a Cauchy sequence $(x_n)_{n=1}^\infty$ such that $(f(x_n))_{n=1}^\infty$ is not Cauchy in Y.

6. If (X, d_X) and (Y, d_Y) are metric spaces and $f\colon X \to Y$ is a function, then f is called a Lipschitz map if there is a constant $C > 0$ so that $d_Y(f(x_1), f(x_2)) \le C d_X(x_1, x_2)$.

   Show that Lipschitz maps (from one metric space to another) are uniformly continuous.

7. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $X = Y = \mathbb{K}^n$ and take the metrics $d_X(x_1, x_2) = \|x_1 - x_2\|_2$ on X, $d_Y(y_1, y_2) = \|y_1 - y_2\|_1$ on Y. Let $f\colon X \to Y$ be the map given by $f(x) = x$.

   Show that $f\colon X \to Y$ is a Lipschitz map and that the inverse map $f^{-1}\colon Y \to X$ is also Lipschitz.

8. Let $X \subset \mathbb{R}^2$ be a subset of the plane with nonempty interior. (We take the usual euclidean metric on $\mathbb{R}^2$.) Show that X must be of second category in $\mathbb{R}^2$.

   [Hint: Show that X contains a closed ball.]