Mathematics 321 2008–09
Exercises 1
[Due Friday October 17th.]

1. In a metric space (X, d) show that an open ball $B(x_0, r)$ (for $x_0 \in X$, $r > 0$) is an open set.

2. In a metric space (X, d) show that a closed ball $\bar{B}(x_0, r)$ (for $x_0 \in X$, $r > 0$) is a closed set.

3. Consider the set $\mathbb{N}$ as a submetric space of the real line (with the usual distance on $\mathbb{R}$). Show that the ball $B(n, 1)$ of radius 1 about every point $n \in \mathbb{N}$ is open. Deduce that every subset of $\mathbb{N}$ is open.

4. Consider the set $\mathbb{N}$ with the usual metric again. Show that the closed ball $\bar{B}(5, 1) = \{4, 5, 6\}$ but that the closure of the open ball $B(5, 1)$ is different.

5. Let (X, d) be a metric space and $x_0 \in X$ a given point. Define $f \colon X \to \mathbb{R}$ by $f(x) = d(x, x_0)$. Show that f is a continuous function.

   [Hint: Show that in a metric space (X, d) we have $|d(x, z) - d(y, z)| \leq d(x, y)$ for $x, y, z \in X$.]

6. In the metric space $(\mathbb{R}^2, d)$, where d is the standard euclidean metric, show that the square $\{(x, y) : |x| < 1, |y| < 1\}$ is open.

7. Consider the two point space $X = \{1, 2\}$ (with the discrete topology)[1] and the normed space $CB(X, \mathbb{K})$ with the supremum norm $\|f\|_\infty = \max(|f(1)|, |f(2)|)$. (Here $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.)

   Show that $CB(X, \mathbb{K})$ is two-dimensional by exhibiting a vector space isomorphism with $\mathbb{K}^2$.

   Calculate the distance $d_\infty(f, g) = \|f - g\|_\infty$ on $CB(X, \mathbb{K})$ in terms of the values of $f, g \in CB(X, \mathbb{K})$.

   What is the open ball of radius 1 around the origin in $CB(X, \mathbb{K})$? (We usually call this the 'unit ball' for the normed space.)

[1] The discrete topology on any set Y arises from the metric ρ on Y given by $\rho(y, z) = 1$ if $y \neq z$ and $\rho(y, y) = 0$ $(y, z \in Y)$. In the case of $\{1, 2\}$ this just happens to be the same as the same as the standard euclidean value distance.