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Chapter 5: Application: Fourier Series

For lack of time, this chapter is only an outline of some applications of Functional Analysis and some proofs are not complete.

5.1 Definition. If $f \in L^1[0, 2\pi]$, then the *Fourier coefficients* of f are

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t)\, dt \qquad (n \in \mathbb{Z}).$$

The *Fourier series* of f is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

5.2 Remark. Using the fact that the functions $\phi_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ $(n \in \mathbb{Z})$ form an orthonormal basis for $L^2[0, 2\pi]$ we can conclude that, for $f \in L^2[0, 2\pi]$,

$$f = \sum_{n \in \mathbb{Z}} \langle f, \phi_n \rangle \phi_n$$

independent of the order of the ϕ_n and with convergence in the norm of $L^2[0, 2\pi]$. See example 4.11. As $\langle f, \phi_n \rangle \phi_n(t) = \hat{f}(n) e^{int}$, we see that the Fourier series of a function $f \in L^2[0, 2\pi]$ converges to f in L^2 norm.

The coefficients $\hat{f}(n)$ can be defined for $f \in L^1[0, 2\pi]$ (recall $L^2[0, 2\pi] \subset L^1[0, 2\pi]$), but there is no guarantee that the series converges to the function in any sense. However it is true that the Fourier series (or the sequence of Fourier coefficients $\hat{f}(n)$, $n \in \mathbb{Z}$) uniquely determines the function $f \in L^1[0, 2\pi]$ (see Theorem 5.8) even though the way the Fourier series of $f \in L^1[0, 2\pi]$ determines the function f is not very straightforward.

5.3 Theorem (Riemann-Lebesgue Lemma)**.** *If* $f \in L^1[0, 2\pi]$*, then*

$$\lim_{|n| \to \infty} \hat{f}(n) = 0.$$

Proof. Some of the details are omitted here, but the idea is based on the observation that the result is quite obvious when f is a trigonometric polynomial. These polynomials are dense in $L^1[0, 2\pi]$ (and each trigonometric polynomial p has only finitely many nonzero Fourier coefficients, hence $\hat{p}(n) = 0$ for $|n|$ large).

We can define a linear operator

$$\begin{aligned} T\colon L^1[0,2\pi] &\to \ell^\infty \\ T(f) &= (\hat{f}(0), \hat{f}(1), \hat{f}(-1), \hat{f}(2), \hat{f}(-2), \ldots). \end{aligned}$$

To check that $T(f) \in \ell^\infty$ when $f \in L^1[0, 2\pi]$ use the fact that (for each $n \in \mathbb{Z}$)

$$\begin{aligned} |\hat{f}(n)| &\leq \frac{1}{2\pi} \left| \int_0^{2\pi} e^{-int} f(t)\, dt \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)|\, dt \\ &\quad \text{using } |e^{-int}| = 1 \\ &= \frac{1}{2\pi} \|f\|_1. \end{aligned}$$

So $\|T(f)\|_\infty = \sup_n |\hat{f}(n)| < \infty$ and $T(f) \in \ell^\infty$. By linearity of the integral, we can see that T is linear and then the inequality $\|T(f)\|_\infty \leq (1/2\pi)\|f\|_1$ shows that T is a bounded operator.

As T maps the trigonometric polynomials into c_0, and $c_0 \subset \ell^\infty$ is closed, it follows (using density of the trigonometric polynomials in $L^1[0, 2\pi]$) that the values of T are all in c_0. [The idea is that if $f \in L^1[0, 2\pi]$, there is a sequence $(p_n)_{n=1}^\infty$ of trigonometric polynomials with $\lim_{n\to\infty} p_n = f$ (in the norm $\|\cdot\|_1$). By continuity of T, $T(f) = \lim_{n\to\infty} T(p_n)$ in ℓ^∞. As $T(p_n) \in c_0$ for each n, and c_0 is closed in ℓ^∞, we must have $t(f) \in c_0$.]

We have not proved that the trigonometric polynomials are actually dense in $L^1[0, 2\pi]$. The proof of that is not so different to the proof that the same trigonometric polynomials are dense in $L^2[0, 2\pi]$. Some indication of how a proof can go was in Example 4.11. □

5.4 Corollary. *The map* $T\colon L^1[0, 2\pi] \to c_0$ *given by*

$$T(f) = (\hat{f}(0), \hat{f}(1), \hat{f}(-1), \hat{f}(2), \hat{f}(-2), \ldots)$$

is a bounded linear operator.

The corollary is essentially a restatement of the Riemann-Lebesgue Lemma, or of the way we proved it.

5.5 Definition. The *Dirichlet kernels* are

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt} \qquad (n = 0, 1, 2, \ldots).$$

5.6 Remark. This is related to Fourier series because the partial sums of the Fourier series of $f \in L^1[0, 2\pi]$ are given by

$$\sum_{j=-n}^{n} \hat{f}(j) e^{ijt} = \frac{1}{2\pi} \int_0^{2\pi} D_n(t-\theta) f(\theta)\, d\theta.$$

This formula is easy to verify and the integral involved is known as a *convolution* (of f and D_n).

We sometimes write $S_n f$ for the function

$$(S_n f)(t) = \sum_{j=-n}^{n} \hat{f}(j) e^{ijt}$$

and refer to $S_n f$ as the nth *partial sum* of the Fourier series of f. The *partial sum operator* is a linear operator on functions. We can say $S_n \colon L^1[0, 2\pi] \to L^1[0, 2\pi]$ or we can regard S_n as having its values in a nicer space than $L^1[0, 2\pi]$, such as the 2π-periodic continuous functions on $[0, 2\pi]$.

5.7 Lemma.

$$\|D_n\|_1 \to \infty \text{ as } n \to \infty.$$

This is proved by a fairly direct calculation to show that

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2} t}$$

and then estimating the integral in a somewhat careful way.

5.8 Theorem. *The map T of Corollary 5.4 is injective but not surjective.*

This Theorem implies that the Fourier coefficients of a function $f \in L^1[0, 2\pi]$ determine f completely, but that the Riemann-Lebesgue lemma is not a full description of the possible sequences of Fourier coefficients for such f.

The proof of injectivity relies on Lusin's theorem and the Weierstrass theorem to show that if $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then

$$\int_E f(t)\,dt = 0 \tag{1}$$

for all measurable subsets $E \subset [0, 2\pi]$. From $\hat{f}(n) = 0$ for all n we quickly see that $\int_0^{2\pi} f(t)p(t)\,dt = 0$ for all trigonometric polynomials p and then from the Weierstrass theorem it follows easily that $\int_0^{2\pi} f(t)g(t)\,dt = 0$ for all continuous 2π-periodic functions g. From Lusin's theorem we can get a sequence g_n of such continuous 2π-periodic functions so that

$$\lim_{m\to\infty} g_n(t) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E \end{cases} \qquad \text{for almost every } t \in [0, 2\pi].$$

Moreover we can get such a sequence where $\sup_t |g_n(t)| = \|g_n\|_\infty \leq 1$ for all n. By the Lebesgue dominated convergence theorem we get (1).

If f is not zero in $L^1[0, 2\pi]$ then it must have nonzero real or imaginary part. Then we can find $\delta = 1/k > 0$ so that one of the following sets E has positive measure:

$$\begin{aligned} E &= \{t \in [0, 2\pi] : \Re f(t) > \delta\} \\ E &= \{t \in [0, 2\pi] : \Re f(t) < -\delta\} \\ E &= \{t \in [0, 2\pi] : \Im f(t) > \delta\} \\ E &= \{t \in [0, 2\pi] : \Im f(t) < -\delta\} \end{aligned}$$

In all cases equation (1) leads to a contradiction.

This way we can show that T is injective.

If it was surjective then it would be a bijective bounded linear operator between Banach spaces and so the open mapping theorem would say that its inverse T^{-1} would have to be bounded from c_0 to $L^1[0, 2\pi]$. But that is not the case because from Lemma 5.7 one can see $\|TD_n\| = 1$ in c_0 but $\|T^{-1}(TD_n)\|_1 = \|D_n\|_1 \to \infty$ as $n \to \infty$.

The next result aims to show that Fourier series are not so well-behaved even for continuous 2π-periodic functions.

5.9 Theorem. *There exists a continuous 2π-periodic function f such that the partial sums of its Fourier series do not converge at $t = 0$, that is such that*

$$\lim_{n\to\infty} (S_n f)(0)$$

does not exist.

Proof. This can be shown via the uniform boundedness principle applied to the Banach space $CP[0, 2\pi]$ of all continuous functions $f: [0, 2\pi] \to \mathbb{C}$ with $f(0) = f(2\pi)$. (These functions are the restrictions to $[0, 2\pi]$ of continuous 2π-periodic functions $f: \mathbb{R} \to \mathbb{C}$. It is quite easy to see that $CP[0, 2\pi] = \{f \in C[0, 2\pi] : f(0) - f(1) = 0\}$ is a closed linear subspace of $C[0, 2\pi]$. It is the kernel of the bounded linear functional $\alpha: C[0, 2\pi] \to \mathbb{C}$ given by $\alpha(f) = f(0) - f(1)$. So $CP[0, 2\pi]$ is a Banach space in the norm $\|\cdot\|_\infty$ we usually use on $C[0, 2\pi]$.) We take the supremum norm $\|f\| = \sup_{t\in[0,2\pi]} |f(t)|$ on $CP[0, 2\pi]$ and then it is a Banach space.

For each n the linear operator

$$\begin{aligned} s_n: CP[0, 2\pi] &\to \mathbb{C} \\ f &\mapsto (S_n f)(0) = \frac{1}{2\pi}\int_0^{2\pi} f(t) D_n(t)\, dt \end{aligned}$$

can be seen to have norm $\|s_n\|_{op} = \frac{1}{2\pi}\|D_n\|_1 < \infty$. (This requires some proof, but is not very difficult. To show $\|s_n\|_{op} \leq \frac{1}{2\pi}\|D_n\|_1$, estimate $|s_n(f)|$ by taking the absolute value inside the integral. To show that $\|s_n\| \geq \frac{1}{2\pi}\|D_n\|_1$, take $f(t) = \overline{D_n(t)}/|D_n(t)|$ and check that $f \in CP[0, 2\pi]$, $\|f\|_\infty = 1$ and $s_n(f) = \frac{1}{2\pi}\|D_n\|_1$.)

If $\lim_{n\to\infty} (S_n f)(0)$ did exist for each $f \in CP[0, 2\pi]$, then it would also be true that

$$\sup_{n\geq 0} |(S_n f)(0)| < \infty$$

for each $f \in CP[0, 2\pi]$. By the uniform boundedness principle it would then follow that $\sup_{n\geq 0} \|s_n\|_{op} < \infty$. But this is false and so there must exist f as required. □

This type of proof is not constructive. It does not tell you how to find a function $f \in CP[0, 2\pi]$ with a Fourier series that fails to converge at $t = 0$. It only tells you that such a function exists.

By the way, there is nothing very special about $t = 0$. A similar proof would show that for any $\theta \in [0, 2\pi]$ there is a function $f \in CP[0, 2\pi]$ such that $\lim_{n\to\infty}(S_n f)(\theta)$ fails to exist. Actually this also follows from the theorem by making a change of variables.

To see the details of these results, look in W. Rudin, *Real and Complex Analysis*.

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