321 2008–09

Chapter 1: Banach Spaces

The main motivation for functional analysis was probably the desire to understand solutions of differential equations. As with other contexts (such as linear algebra where the study of systems of linear equations leads us to vector spaces and linear transformations) it is useful to study the properties of the set or space where we seek the solutions and then to cast the left hand side of the equation as an operator or transform (from a space to itself or to another space). In the case of a differential equation like

$$\frac{dy}{dx} - y = 0$$

we want a solution to be a continuous function $y = y(x)$, or really a differentiable function $y = y(x)$. For partial differential equations we would be looking for functions $y = y(x_1, x_2, \ldots, x_n)$ on some domain in $\mathbb{R}^n$ perhaps.

The ideas involve considering a suitable space of functions, considering the equation as defining an operator on functions and perhaps using limits of some kind of 'approximate solutions'. For instance in the simple example above we might define an operation $y \mapsto L(y)$ on functions where

$$L(y) = \frac{dy}{dx} - y$$

and try to develop properties of the operator so as to understand solutions of the equation, or of equations like the original. One of the difficulties is to find a good space to use. If y is differentiable (which we seem to need to define $L(y)$) then $L(y)$ might not be differentiable, maybe not even continuous.

It is not our goal to study differential equations or partial differential equations in this module (321). We will study functional analysis largely for its own sake. An analogy might be a module in linear algebra without most of the many applications. We will touch on some topics like Fourier series that are illuminated by the theories we consider, and may perhaps be considered as subfields of functional analysis, but can also be viewed as important for themselves and important for many application areas.

Some of the more difficult problems are nonlinear problems (for example nonlinear partial differential equations) but our considerations will be restricted to linear operators. This is partly because the nonlinear theory is complicated and rather fragmented, maybe you could say it is underdeveloped, but one can argue that linear approximations are often used for considering nonlinear problems. So, one relies on the fact that the linear problems are relatively tractable, and on the theory we will consider.

The main extra ingredients compared to linear algebra will be that we will have a norm (or length function for vectors) on our vector spaces and we will also be concerned mainly with infinite dimensional spaces.

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1.1 Normed spaces

1.1.1 Notation. We use $\mathbb{K}$ to stand for either one of $\mathbb{R}$ or $\mathbb{C}$. In this way we can develop the theory in parallel for the real and complex scalars.

We mean however, that the choice is made at the start of any discussion and, for example when we ask that E and F are vector spaces over $\mathbb{K}$ we mean that the same $\mathbb{K}$ is in effect for both.

1.1.2 Definition. A *norm* on a vector space E over the field $\mathbb{K}$ is a function $x \mapsto \|x\|: E \to [0, \infty) \subseteq \mathbb{R}$ which satisfies the following properties

(i) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ (all $x, y \in E$);

(ii) (scaling property) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, $x \in E$;

(iii) $\|x\| = 0 \Rightarrow x = 0$ (for $x \in E$).

A vector space E over $\mathbb{K}$ together with a chosen norm $\|\cdot\|$ is called a *normed space* (over $\mathbb{K}$) and we write $(E, \|\cdot\|)$.

A *seminorm* is like a norm except that it does not satisfy the last property (nonzero elements can have length 0). Rather than use the notation $\|\cdot\|$ we use $p\colon E \to [0, \infty)$ for a seminorm. Then we insist that a seminorm satisfies the triangle inequality ($p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$) and the property about scaling ($p(\lambda x) = |\lambda| p(x)$ for $x \in E$ and $\lambda \in \mathbb{K}$). We will use seminorms fairly rarely in this module, though there are contexts in which they are very much used.

1.1.3 Examples. The most familiar examples of normed spaces are $\mathbb{R}^n$ and $\mathbb{C}^n$. The fact that the norms do in fact satisfy the triangle inequality is not entirely obvious (usually proved via the Cauchy Schwarz inequality) but we will take that as known for now. Later we will prove something more general.

- $E = \mathbb{R}^n$ with $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{\sum_{i=1}^n x_i^2}$ is a normed space (over the field $\mathbb{R}$). We understand the vector space operations to be the standard (coordinatewise defined) ones.

- $E = \mathbb{C}^n$ with $\|(z_1, z_2, \ldots, z_n)\| = \sqrt{\sum_{j=1}^n |z_j|^2}$ is a normed space (over the field $\mathbb{C}$).

- In both cases, we may refer to the above norms as $\|\cdot\|_2$, as there are other possible norms on $\mathbb{K}^n$. An example is given by

$$\|(x_1, x_2, \ldots, x_n)\|_1 = \sum_{j=1}^n |x_j|.$$

  Even though it is not as often used as the standard (Euclidean) norm $\|\cdot\|_2$, it is much easier to verify that $\|\cdot\|_1$ is a norm on $\mathbb{K}^n$ than it is to show $\|\cdot\|_2$ is a norm.

1.2 Metric spaces

This subsection may be largely review of material from module 221 apart from Lemma 1.2.7 below.

1.2.1 Definition. If X is a set and $d: X \times X \to [0, \infty) \subset \mathbb{R}$ is a function with the properties:

(i) $d(x, y) \geq 0$ (for $x, y \in X$);

(ii) $d(x, y) = d(y, x)$ (for $x, y \in X$);

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality) for $x, yz, zinX$;

(iv) $x, y \in X$ and $d(x, y) = 0 \to x = y$,

we say d is a metric, and the combination (X, d) is called a *metric space*.

If we omit the last condition that $d(x, y) = 0$ implies $x = y$, we call d a *pseudometric* or *semimetric*.

1.2.2 Notation. In any metric space (X, d) we define *open balls* as follows. Fix any point $x_0 \in X$ (which we think of as the centre) and any $r > 0$. Then the *open ball* of radius r centre x_0 is

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

The *closed ball* with the same centre and radius is

$$\bar{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}.$$

1.2.3 Open and closed subsets. A set $G \subseteq X$ (where we now understand that (X, d) is a particular metric space) is *open* if each $x \in G$ is an *interior point* of G.

A point $x \in G$ is called an interior point of G if there is a ball $B(x, r) \subset G$ with $r > 0$.

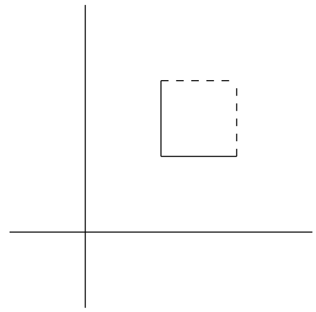
Picture for an open set: G contains **none** of its 'boundary' points.

Any union $G = \bigcup_{i \in I} G_i$ of open sets $G_i \subseteq X$ is open (I any index set, arbitrarily large).

$F \subseteq X$ is *closed* if its complement $X \setminus F$ is open.

Picture for a closed set: F contains **all** of its 'boundary' points.

Note that open and closed are opposite extremes. There are plenty of sets which are neither open nor closed. For example $\{z = x + iy \in \mathbb{C} : 1 \leq x, y < 2\}$ is a square in the plane $\mathbb{C} = \mathbb{R}^2$ with some of the 'boundary' included and some not. It is <u>neither</u> open nor closed.



Any intersection $F = \bigcup_{i\in I} F_i$ of closed sets $F_i \subset X$ is closed.
Finite intersection $G_1 \cap G_2 \cap \cdots \cap G_n$ of open sets are open.
Finite unions of closed sets are closed.

1.2.4 Exercise. Show that an open ball $B(x_0, r)$ in a metric space (X, d) is an open set.

1.2.5 Interiors and closures. Fix a metric space (X, d).
For any set $E \subseteq X$, the interior E° is the set of all its interior points.

$$E^\circ = \{x \in E : \exists r > 0 \text{ with } B(z, r) \subseteq E\}$$

is the largest open subset of X contained in E. Also

$$E^\circ = \bigcup\{G : G \subseteq E, G \text{ open in } \mathbb{C}\}$$

Picture: E° is E minus all its 'boundary' points.
The closure of E is

$$\bar{E} = \bigcap\{F : F \subset X, E \subset F \text{ and } F \text{ closed}\}$$

and it is the smallest closed subset of X containing E.
Picture: $\bar{E}$ is E with all its 'boundary' points added.
Properties: $\bar{E} = X \setminus (X \setminus E)^\circ$ and $E^\circ = X \setminus (\overline{X \setminus E})$.

1.2.6 Boundary. Again we assume we have a fixed metric space (X, d) in which we work.
The *boundary* ∂E of a set $E \subseteq X$ is defined as $\partial E = \bar{E} \setminus E^\circ$.
This formal definition makes the previous informal pictures into facts.

1.2.7 Lemma. *On any normed space* $(E, \|\cdot\|)$ *we can define a metric via* $d(x, y) = \|x - y\|$.

From the metric we also get a topology (notion of open set).

In a similar way a seminorm p *on* E *gives rise to a pseudo metric* $\rho(x, y) = p(x - y)$ *(like a metric but* $\rho(x, y) = 0$ *is allowed for* $x \neq y$*). From a pseudo metric, we get a (non Hausdorff) topology by saying that a set is open if it contains a ball* $B_\rho(x_0, r) = \{x \in E : \rho(x, x_0) < r\}$ *of some positive radius* $r > 0$ *about each of its points.*

Proof. It is easy to check that d as defined satisfies the properties for a metric.

- $d(x, y) = \|x - y\| \in [0, \infty)$
- $d(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = d(y, x)$
- $d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$
- $d(x, y) = 0 \Rightarrow \|x - y\| = 0 \Rightarrow x - y = 0 \Rightarrow x = y$.

The fact that pseudo metrics give rise to a topology is quite easy to verify.

□

1.2.8 Continuity. Let (X, d_X) and (Y, d_Y) be two metric spaces.

If $f: X \to Y$ is a function, then f is called *continuous at a point* $x_0 \in X$ if for each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that

$$x \in X,\ d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$$

$f: X \to Y$ is called *continuous* if it is continuous at each point $x_0 \in X$.

1.2.9 Example. If (X, d) is a metric space and $f: X \to \mathbb{R}$ is a function, then when we say f is continuous we mean that it is continuous from the metric space X to the metric space $\mathbb{R} = \mathbb{R}$ with the normal absolute value metric.

Similarly for complex valued functions $f: X \to \mathbb{C}$, we normally think of continuity to mean the situation where $\mathbb{C}$ has the usual metric.

1.2.10 Proposition. *If* $f: X \to Y$ *is a function between two metric spaces* X *and* Y*, then* f *is continuous if and only if it satisfies the following condition: for each open set* $U \subset Y$*, its inverse image* $f^{-1}(U) = \{x \in X : f(x) \in U\}$ *is open in* X*.*

Proof. Exercise. □

1.2.11 Limits. We now define limits of sequences in a metric space (X, d). A sequence $(x_n)_{n=1}^{\infty}$ in X is actually a function $x\colon \mathbb{N} \to X$ from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ to X where, by convention, we use the notation x_n instead of the usual function notation $x(n)$.

To say $\lim_{n\to\infty} x_n = \ell$ (with $\ell \in X$ also) means:

for each $\varepsilon > 0$ it is possible to find $N \in \mathbb{N}$ so that

$$n \in \mathbb{N}, n > N \Rightarrow d(x_n, \ell) < \varepsilon.$$

An important property of limits of sequences in metric spaces is that a sequence can have at most one limit. In a way we have almost implicitly assumed that by writing $\lim_{n\to\infty} x_n$ as though it is one thing. Notice however that there are sequences with no limit.

1.2.12 Proposition. *Let X be a metric space, $S \subset X$ and $x \in X$. Then $x \in \bar{S} \iff$ there exists a sequence $(s_n)_{n=1}^{\infty}$ with $s_n \in S$ for all n and $\lim_{n\to\infty} s_n = x$ (in X).*

Proof. Not given here. (Exercise.) □

1.2.13 Proposition. *Let X and Y be two metric spaces. If $f\colon X \to Y$ is a function and $x_0 \in X$ is a point, then f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(z_0)$ holds for <u>all</u> sequences $(x_n)_{n=1}^{\infty}$ in X with $\lim_{n\to\infty} x_n = x_0$.*

Proof. Not given here. (Exercise.) □

1.2.14 Remark. Consider the case where we have sequences in $\mathbb{R}$, which is not only a metric space but where we can add and multiply.

One can show that the limit of a sum is the sum of the limits (provided the individual limits make sense). More symbolically,

$$\lim_{n\to\infty} x_n + y_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

Similarly

$$\lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

if both individual limits exist.

We also have the result on limits of quotients,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

provided $\lim_{n\to\infty} y_n \neq 0$. In short the limit of a quotient is the quotient of the limits provided the limit in the denominator is not zero.

One can use these facts to show that sums and products of continuous $\mathbb{R}$-valued functions on metric spaces are continuous. Quotients also if no division by 0 occurs.

1.2.15 Definition. If X is a set then a *topology* $\mathcal{T}$ on X is a collection of subsets of X with the following properties

(i) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$;

(ii) if $U_i \in \mathcal{T}$ for all $i \in I =$ some index set, then $\bigcup_{i\in I} U_i \in \mathcal{T}$;

(iii) if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

A set X together with a topology $\mathcal{T}$ on X is called a *topological space* $(X, \mathcal{T})$.

1.2.16 Remark. Normally, when we consider a topological space $(X, \mathcal{T})$, we refer to the subsets of X that are in $\mathcal{T}$ as *open subsets* of X.

We should perhaps explain immediately that if we start with a metric space (X, d) and if we take $\mathcal{T}$ to be the open subsets of (X, d) (according to the definition we gave earlier), then we get a topology $\mathcal{T}$ on X.

Notice that at least some of the concepts we had for metric spaces can be expressed using only open sets without the necessity to refer to distances.

- $F \subset X$ is closed $\iff$ $X \setminus F$ is open
- $f \colon X \to Y$ is continuous $\iff$ $f^{-1}(U)$ is open in X whenever U is open in Y.

1.2.17 Example. (i) One example of a topology on any set X is the topology $\mathcal{T} = \mathcal{P}(X) =$ the power set of X (all subsets of X are in $\mathcal{T}$, all subsets declared to be open).

We can also get to this topology from a metric, where we define

$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

In this metric the open ball of radius $1/2$ about any point $x_0 \in X$ is

$$B(x_0, 1/2) = \{x_0\}$$

and all one points sets are then open. As unions of open sets are open, it follows that all subsets are open.

The metric is called the discrete metric and the topology is called the discrete topology.

All functions $f\colon X \to Y$ will be continuous if X has the discrete topology (and Y can have any valid topology).

(ii) The other extreme is to take (say when X has at least 2 elements) $\mathcal{T} = \{\emptyset, X\}$. This is a valid topology, called the indiscrete topology.

If X has at least two points $x_1 \neq x_2$, there can be no metric on X that gives rise to this topology. If we thought for a moment we had such a metric d, we can take $r = d(x_1, x_2)/2$ and get an open ball $B(x_1, r)$ in X that contains x_1 but not x_2. As open balls in metric spaces are in fact open subsets, we must have $B(x_1, r)$ different from the empty set and different from X.

The only functions $f\colon X \to \mathbb{R}$ that are continuous are the constant functions in this example. On the other hand every function $g\colon Y \to X$ is continuous (no matter what Y is, as long as it is a topological space so that we can say what continuity means).

This example shows that there are topologies that do not come from metrics, or topological spaces where there is no metric around that would give the same idea of open set. Or, in other language, topological spaces that do not arise from metric spaces (are not metric spaces). Our example is not very convincing, however. It seems very silly, perhaps. If we studied topological spaces in a bit more detail we would come across more significant examples of topological spaces that are not metric spaces (and where the topology does not arise from any metric).

1.2.18 Compactness. Let (X, d) be a metric space.

Let $T \subseteq X$. An *open cover* of T is a family $\mathcal{U}$ of open subsets of X such that

$$T \subseteq \bigcup \{U : U \in \mathcal{U}\}$$

A subfamily $\mathcal{V} \subseteq \mathcal{U}$ is called a subcover of $\mathcal{U}$ if $\mathcal{V}$ is also a cover of T.

T is called *compact* if each open cover of T has a finite subcover.

T is called *bounded* if there exists $R \geq 0$ and $x_0 \in X$ with $T \subseteq \bar{B}(x_0, R)$.

One way to state the *Heine-Borel theorem* is that a subset $T \subseteq \mathbb{R}^n$ is compact if and only if it is both (1) closed and (2) bounded.

Continuous images of compact sets are compact: $T \subseteq X$ compact, $f\colon T \to Y$ continuous implies $f(T)$ compact.

1.2.19 Definition. If (X, d) is a metric space, then a subset $T \subset X$ is called *sequentially compact* if it has the following property:

> Each sequence $(t_n)_{n=1}^\infty$ has a subsequence $(t_{n_j})_{j=1}^\infty$ with $\lim_{j\to\infty} t_{n_j} = \ell \in T$ for some $\ell \in T$.

In words, every sequence in T has a subsequence with a limit in T.

1.2.20 Theorem. *In a metric space (X, d) a subset $T \subset X$ is compact if and only if it is sequentially compact.*

Proof. Omitted here. □

1.2.21 Remark. We can abstract almost all of the above statements about compactness to general topological spaces rather than a metric space. Metric spaces are closer to what we are familiar with, points in space or the plane or the line, where we think we can see geometrically what distance means (straight line distance between points).

At least in normed spaces, many familiar ideas still work in some form.

One thing to be aware of is that sequences are not as useful in topological spaces as they are in metric spaces. Sequences in a topological space may converge to more than one limit. Compactness (defined via open covers) is not the same as sequential compactness in every topological space. Sequences do not always describe closures or continuity as they do in metric spaces (Propositions 1.2.12 and 1.2.13).

1.3 Examples of normed spaces

1.3.1 Examples. (i) $\mathbb{K}^n$ with the standard Euclidean norm is complete.

(ii) If X is a metric space (or a topological space) we can define a norm on $E = BC(X) = \{f\colon X \to \mathbb{K} : f \text{ bounded and continuous}\}$ by

$$\|f\| = \sup_{x\in X} |f(x)|.$$

To be more precise, we have to have a vector space before we can have a norm. We define the vector space operations on $BC(X)$ in the 'obvious' (pointwise) way. Here are the definition of $f + g$ and λf for $f, g \in BC(X)$, $\lambda \in \mathbb{K}$:

- $(f+g)(x) = f(x) + g(x)$ (for $x \in X$)
- $(\lambda f)(x) = \lambda(f(x)) = \lambda f(x)$ (for $x \in X$)

We should check that $f+g, \lambda f \in BC(X)$ always and that the vector space rules are satisfied, but we leave this as an exercise if you have not seen it before.

It is not difficult to check that we have defined a norm on $BC(X)$. It is known often as the 'uniform norm' or the 'sup norm' (on X).

(iii) If we replace X by a compact Hausdorff space K in the previous example, we know that every continuous $f\colon K \to \mathbb{K}$ is automatically bounded. The usual notation then is to use $C(K)$ rather than $BC(K)$.

Otherwise everything is the same (vector space operations, supremum norm).

(iv) If we take for X the discrete space $X = \mathbb{N}$, we can consider the example $BC(\mathbb{N})$ as a space of functions on $\mathbb{N}$ (with values in $\mathbb{K}$). However, it is more usual to think in terms of this example as a space of sequences. The usual notation for it is

$$\ell^\infty = \{(x_n)_{n=1}^\infty : x_n \in \mathbb{K} \forall n \text{ and } \sup_n |x_n| < \infty\}.$$

So ℓ^∞ is the space of all *bounded (infinite) sequences of scalars*. The vector space operations on sequences are defined as for functions (pointwise or term-by-term)

$$\begin{aligned} (x_n)_{n=1}^\infty + (y_n)_{n=1}^\infty &= (x_n + y_n)_{n=1}^\infty \\ \lambda (x_n)_{n=1}^\infty &= (\lambda x_n)_{n=1}^\infty \end{aligned}$$

and the uniform or supremum norm on ℓ^∞ is typically denoted by a subscript ∞ (to distinguish it from other norms on other sequence spaces that we will come to soon).

$$\|(x_n)_{n=1}^\infty\|_\infty = \sup_n |x_n|.$$

It will be important for us to deal with complete normed spaces (which are called Banach spaces). First we will review some facts about complete metric spaces and completions. A deeper consequence of completeness is the Baire category theorem.

Completeness is important if one wants to prove that equations have a solution, one technique is to produce a sequence of approximate solutions. If the space is not complete (like $\mathbb{Q}$, the rationals) the limit of the sequence may not be in the space we consider. (For instance in $\mathbb{Q}$ one could find a sequence approximating solution of $x^2 = 2$, but the limit $\sqrt{2}$ would not be in $\mathbb{Q}$.)

1.4 Complete metric spaces

This is also largely review.

1.4.1 Definition. If (X, d) is a metric space, then a sequence $(x_n)_{n=1}^{\infty}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$ it is possible to find N so that

$$n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon.$$

1.4.2 Remark. The definition of a Cauchy sequence requires a metric and not just a topology on X.

There is a more abstract setting of a 'uniformity' on X where it makes sense to talk about Cauchy sequences (or Cauchy nets). We will not discuss this generalisation.

1.4.3 Proposition. *Every convergent sequence in a metric space (X, d) is a Cauchy sequence.*

Proof. We leave this as an exercise.

The idea is that a Cauchy sequence is one where, eventually, all the remaining terms are close to one another. A convergent sequence is one where, eventually, all the remaining terms are close to the limit. If they are close to the same limit then they are also close to one another. □

1.4.4 Definition. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges (to some limit in X).

1.4.5 Example. The rationals $\mathbb{Q}$ with the usual (absolute value) metric is not complete. There are sequences in $\mathbb{Q}$ that converge to irrational limits (like $\sqrt{2}$). Such a sequence will be Cauchy in $\mathbb{R}$, hence Cauchy in $\mathbb{Q}$, but will not have a limit in $\mathbb{Q}$.

1.4.6 Proposition. *If (X, d) is a complete metric space and $Y \subseteq X$ is a subset, let d_Y be the metric d restricted to Y.*

Then the (submetric space) (Y, d_Y) is complete if and only if Y is closed in X.

Proof. Suppose first (Y, d_Y) is complete. If x_0 is a point of the closure of Y in X, then there is a sequence $(y_n)_{n=1}^\infty$ of points $y_n \in Y$ that converges (in X) to x_0. The sequence $(y_n)_{n=1}^\infty$ is then Cauchy in (X, d). But the Cauchy condition involves only distances $d(y_n, y_m) = d_Y(y_n, y_m)$ between the terms and so $(y_n)_{n=1}^\infty$ is Cauchy in Y. By completeness there is $y_0 \in Y$ so that $y_n \to y_0$ as $n \to \infty$. That means $\lim_{n\to\infty} d_Y(y_n, y_0) = 0$ and that is the same as $\lim_{n\to\infty} d(y_n, y_0) = 0$ or $y_n \to y_o$ when we consider the sequence and the limit in X. Since also $y_n \to x_0$ as $n \to \infty$, and limits in X are unique, we conclude $x_0 = y_0 \in Y$. Thus Y is closed in X.

Conversely, suppose Y is closed in X. To show Y is complete, consider a Cauchy sequence $(y_n)_{n=1}^\infty$ in Y. It is also Cauchy in X. As X is complete the sequence has a limit $x_0 \in X$. But we must have $x_0 \in Y$ because Y is closed in X. So the sequence $(y_n)_{n=1}^\infty$ converges in (Y, d_Y). □

1.4.7 Remark. The following lemma is useful in showing that metric spaces are complete.

1.4.8 Lemma. *Let (X, d) be a metric space in which each Cauchy sequence has a convergent subsequence. Then (X, d) is complete.*

Proof. We leave this as an exercise.

The idea is that as the terms of the whole sequence are eventually all close to one another, and the terms of the convergent subsequence are eventually close to the limit ℓ of the subsequence, the terms of the whole sequence must be eventually close to ℓ. □

1.4.9 Corollary. *Compact metric spaces are complete.*

1.4.10 Definition. If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f : X \to Y$ is called *uniformly continuous* if for each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that

$$x_1, x_2 \in X, d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

1.4.11 Proposition. *Uniformly continuous functions are continuous.*

Proof. We leave this as an exercise.

The idea is that in the ε-δ criterion for continuity, we fix one point (say x_1) as well as $\varepsilon > 0$ and then look for $\delta > 0$. In uniform continuity, the same $\delta > 0$ must work for all $x_1 \in X$. □

1.4.12 Definition. If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f\colon X \to Y$ is called an *isometry* if $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

We could call f *distance preserving* instead of isometric, but the word isometric is more commonly used. Sometimes we consider isometric bijections (which then clearly have isometric inverse maps). If there exists an isometric bijection between two metric spaces X and Y, we can consider them as equivalent metric spaces (because every property defined only in terms of the metric must be shared by Y is X has the property).

1.4.13 Example. Isometric maps are injective and uniformly continuous.

Proof. Let $f\colon X \to Y$ be the map. To show injective, let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then $d_X(x_1, x_2) > 0 \Rightarrow d_Y(f(x_1), f(x_2)) > 0 \Rightarrow f(x_1) \neq f(x_2)$.

To show uniform continuity, take $\delta = \varepsilon$. □

1.4.14 Definition. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces (or (X, d_X) and (Y, d_Y) are metric spaces) then a *homeomorphism* from X onto Y is a bijection $f\colon X \to Y$ with f continuous and f^{-1} continuous.

1.4.15 Remark. If $f\colon X \to Y$ is a homeomorphism of topological spaces, then $V \subset Y$ open implies $U = f^{-1}(V) \subset X$ open (by continuity of f). On the other hand $U \subset X$ open implies $(f^{-1})^{-1}(U) = f(U)$ open by continuity of f^{-1} (since the inverse image of U under the inverse map f^{-1} is the same as the forward image $f(U)$). In this way we can say that

$$U \subset X \text{ is open } \iff f(U) \subset Y \text{ is open}$$

and homeomorphic spaces are identical from the point of view of topological properties.

Note that isometric metric spaces are identical from the point of view of metric properties. The next next result says that completeness transfers between metric spaces that are homeomorphic via a homeomorphism that is uniformly continuous in one direction.

1.4.16 Proposition. *If (X, d_X) and (Y, d_Y) are metric spaces with (X, d_X) complete, and $f\colon X \to Y$ is a homeomorphism with f^{-1} uniformly continuous, then (Y, d_Y) is also complete.*

Proof. Let $(y_n)_{n=1}^{\infty}$ be a Cauchy sequence in Y. Let $x_n = f^{-1}(y_n)$. We claim $(x_n)_{n=1}^{\infty}$ is Cauchy in X. Given $\varepsilon > 0$ find $\delta > 0$ by uniform continuity of f^{-1} so that

$$y, y' \in Y, d_Y(y, y') < \delta \Rightarrow d_X(f^{-1}(y), f^{-1}(y')) < \varepsilon.$$

As $(y_n)_{n=1}^\infty$ is Cauchy in Y, there is $N > 0$ so that

$$n, m \geq N \Rightarrow d_Y(y_n, y_m) < \delta.$$

Combining these, we see

$$n, m \geq N \Rightarrow d_X(f^{-1}(y_n), f^{-1}(y_m)) < \varepsilon.$$

So $(x_n)_{n=1}^\infty$ is Cauchy in X, and so has a limit $x_0 \in X$. By continuity of f at x_0, we get $f(x_n) = y_n \to f(x_0)$ as $n \to \infty$. So $(y_n)_{n=1}^\infty$ converges in Y. This shows that Y is complete. □

1.4.17 Example. There are homeomorphic metric spaces where one is complete and the other is not. For example, $\mathbb{R}$ is homeomorphic to the open unit interval $(0, 1)$.

One way to see this is to take $g: \mathbb{R} \to (0, 1)$ as $g(x) = (1/2) + (1/\pi)\tan^{-1} x$. Another is $g(x) = (1/2) + x/(2(1 + |x|))$.

In the standard absolute value distance $\mathbb{R}$ is complete but $(0, 1)$ is not.

One can use a specific homeomorphism $g: \mathbb{R} \to (0, 1)$ to transfer the distance from $\mathbb{R}$ to $(0, 1)$. Define a new distance on $(0, 1)$ by $\rho(x_1, x_2) = |g^{-1}(x_1) - g^{-1}(x_2)|$. With this distance ρ on $(0, 1)$, the map g becomes an isometry and so $(0, 1)$ is complete in the ρ distance.

The two topologies we get on $(0, 1)$, from the standard metric and from the metric ρ, will be the same. We can see from this example that completeness is not a topological property.

1.4.18 Theorem ((Banach) contraction mapping theorem)**.** *Let (x, d) be a (nonempty) complete metric space and let $f: X \to X$ be a strictly contractive mapping (which means there exists $0 \leq r < 1$ so that $d(f(x_1), f(x_2)) \leq rd(x_1, x_2)$ holds for all $x_1, x_2 \in X$).*

Then f has a unique fixed point in X (that is there is a unique $x \in X$ with $f(x) = x$.

Proof. We omit this proof as we will not use this result. It can be used to show that certain ordinary differential equations have (local) solutions.

The idea of the proof is to start with $x_0 \in X$ arbitrary and to define $x_1 = f(x_0)$, $x_2 = f(x_1)$ *etc.*, that is $x_{n+1} = f(x_n)$ for each n. The contractive property implies that $(x_n)_{n=1}^\infty$ is a Cauchy sequence. The limit $\lim_{n\to\infty} x_n$ is the fixed point x. Uniqueness of the fixed point follows from the contractive property. □

1.5 Completion of a metric space

1.5.1 Theorem. *If (X, d) is a metric space, then there is a complete metric space $(\hat{X}, \hat{d})$ that contains [an isometric copy of] (X, d) as a dense subspace.*

Proof. Let $\hat{X}$ denote the set of equivalence classes of Cauchy sequences $(x_n)_n$ in X where the equivalence relation is that $(x_n)_n \sim (y_n)_n$ if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Then define the distance $\hat{d}(\alpha, \beta)$ between the equivalence class α of $(x_n)_n$ and the equivalence class β of $(y_n)_n$ by

$$\hat{d}(\alpha, \beta) = \lim_{n\to\infty} d(x_n, y_n). \tag{1}$$

There are several things to check. First that $\hat{d}$ is well-defined. To do this we need first to know that the limit in (1) exists. We show $\Big(d(x_n, y_n)\Big)_n$ is Cauchy in $\mathbb{R}$. Let $\varepsilon > 0$ be given. We can find N so that for $n, m > N$ both

$$d(x_n, x_m) < \varepsilon/2 \text{ and } d(y_n, y_m) < \varepsilon/2.$$

Consequently

$$\begin{aligned} d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) - d(x_m, y_m) \\ &= d(x_n, x_m) + d(y_m, y_n) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

The same reasoning with n and m interchanged then shows $d(x_m, y_m) - d(x_n, y_n) \leq \varepsilon$, and so

$$|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon$$

(for $n, m > N$). Thus the limit exists (as $\mathbb{R}$ is complete).

Next we must show that we get the same limit if we take different representatives $(x'_n)_n$ for α and $(y'_n)_n$ for β. This is clear from the inequality

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n)$$

Taking limits gives $\lim d(x'_n, y'_n) \leq \lim d(x_n, y_n)$ and of course the reverse inequality follows in the same way.

Next we must check that $\hat{d}$ is a metric on $\hat{X}$. This is quite straightforward.

To show that there is a copy of (X, d) in $\hat{X}$ is also easy — the (equivalence classes of the) constant sequences

$$x, x, x, \ldots \qquad (x \in X)$$

give the required copy.

To show that this copy is dense in $\hat{X}$, take an arbitrary $\alpha \in \hat{X}$ and let the Cauchy sequence $(x_n)_n$ be a representative for α. Let α_j denote the equivalence class of the constant sequence $x_j, x_j, x_j, \ldots$. From the Cauchy condition it is easy to see that $\hat{d}(\alpha_j, \alpha) \to 0$ as $j \to \infty$.

The final step is to show that $(\hat{X}, \hat{d})$ is complete. Take a Cauchy sequence $(\alpha_j)_j$ in $\hat{X}$. Passing to a subsequence (by observation 1.4.8 it is enough to find a convergent subsequence), we can suppose that

$$\hat{d}(\alpha_j, \alpha_{j+1}) \leq \frac{1}{2^j} \text{ for } j = 1, 2, 3, \ldots$$

Let $(x_{jn})_n = (x_{j1}, x_{j2}, \ldots)$ represent α_j. Choose first N_j so that

$$n, m \geq N_j \Rightarrow d(x_{jn}, x_{jm}) < \frac{1}{j}$$

and also choose N'_j so that

$$n \geq N'_j \Rightarrow d(x_{jn}, x_{j+1\, n}) < \frac{1}{2^j}$$

Then put $n_j = \max(N_j, N'_j)$. By making the n_j larger, if necessary, we can assume that $n_1 < n_2 < n_3 < \cdots$.

Define $y_k = x_{kn_k}$ and let α denote the equivalence class of $(y_n)_n$. We claim that $\alpha_j \to \alpha$ as $j \to \infty$, but first we must verify that $(y_k)_k$ is a Cauchy sequence, so that we can be sure it made sense to refer to its equivalence class.

For $k, \ell > 0$,

$$\begin{aligned}
d(y_k, y_{k+\ell}) &= d(x_{kn_k}, x_{k+\ell\, n_{k+\ell}}) \\
&\leq d(x_{kn_k}, x_{kn_{k+\ell}}) + d(x_{kn_{k+\ell}}, x_{k+\ell\, n_{k+\ell}}) \\
&\leq \frac{1}{k} + \sum_{p=k}^{k+\ell-1} \frac{1}{2^p} \\
&< \frac{1}{k} + \frac{1}{2^{k-1}} \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

This verifies that $(y_k)_k$ is Cauchy.

Finally, to verify that $\alpha_j \to \alpha$, note that for $k \geq n_j$,

$$\begin{array}{rcl} d(x_{jk}, y_k) & = & d(x_{jk}, x_{kn_k}) \\ & \leq & d(x_{jk}, x_{jn_k}) + d(x_{jn_k}, x_{kn_k}) \\ & \leq & \dfrac{1}{j} + \displaystyle\sum_{\ell=j}^{k-1} \dfrac{1}{2^\ell} \\ & \leq & \dfrac{1}{j} + \dfrac{1}{2^{j-1}} \\ \text{Therefore } \hat{d}(\alpha_j, \alpha) & \leq & \dfrac{1}{j} + \dfrac{1}{2^{j-1}} \\ & \to & 0 \text{ as } j \to \infty. \end{array}$$

This finishes the proof that every metric space (X, d) has a completion. □

1.5.2 Remark. To prove that the completion is unique (up to a distance preserving bijection that keeps the copy of X 'fixed') requires just a little more work.

It relies on the fact that a uniformly continuous function on a dense subset X_0 of one metric space X, with values in a complete metric space Y, has a unique continuous extension to a function $: X \to Y$.

1.5.3 Remark. One can also show that the completion of a normed space can be turned into a normed space. This requires defining vector space operations on (equivalence classes) of Cauchy sequences from the normed space. We do this by adding the sequences term by term, and multiplying each term by the scalar. There is checking required — to show that the operations are well defined for equivalence classes. Then we have to define a norm on the completion (which is the distance to the origin in the completion — need to check it satisfies the conditions for a norm and that the norm and distance are related by $\hat{d}(\alpha, \beta) = \|\alpha - \beta\|$. None of the steps are difficult to carry out in detail.

1.6 Baire Category Theorem

1.6.1 Definition. A subset $S \subset X$ of a metric space (X, d) is called *nowhere dense* if the interior of its closure is empty, $(\bar{S})^\circ = \emptyset$.

A subset $E \subset X$ is called of *first category* if it is a countable union of nowhere dense subsets, that is, the union $E = \bigcup_{n=1}^\infty S_n$ of a sequence of nowhere dense sets S_n) $((\bar{S}_n)^\circ = \emptyset \forall n)$.

A subset $Y \subset X$ is called of *second category* if it fails to be of first category.

1.6.2 Example. (a) If a singleton subset $S = \{s\} \subset X$ (X metric) fails to be nowhere dense, then the interior of its closure is not empty. The closure $\bar{S} = S = \{s\}$ and if that has any interior it means it contains a ball of some positive radius $r > 0$. So

$$B_d(s, r) = \{x \in X : d(x, s) < r\} = \{s\}$$

and this means that s is an isolated point of X (no points closer to it than r).

An example where this is possible would be $X = \mathbb{Z}$ with the usual distance (so $B(n, 1) = \{n\}$) and S any singleton subset. Another example is $X = B((2, 0), 1) \cup \{0\} \subset \mathbb{R}^2$ (with the distance on X being the same as the usual distance between points in $\mathbb{R}^2$) and $S = \{0\}$.

(b) In many cases, there are no isolated points in X, and then a one point set is nowhere dense. So a countable subset is then of first category ($S = \{s_1, s_2, \ldots\}$ where the elements can be listed as a finite or infinite sequence).

For example $S = \mathbb{Z}$ is of first category as a subset of $\mathbb{R}$, though it of second category as a subset of itself. $S = \mathbb{Q}$ is of first category both in $\mathbb{R}$ and in itself (because it is countable and points are not isolated).

The idea is that first countable means 'small' in some sense, while second category is 'not small' in the same sense. While it is often not hard to see that a set is of first category, it is harder to see that it fails to be of first category. One has to consider all possible ways of writing the set as a union of a sequence of subsets.

1.6.3 Theorem (Baire Category)**.** *Let (X, d) be a complete metric space which is not empty. Then the whole space $S = X$ is of second category in itself.*

Proof. If not, then X is of first category and that means $X = \bigcup_{n=1}^\infty S_n$ where each S_n is a nowhere dense subset $S_n \subset X$ (with $(\bar{S}_n)^\circ = \emptyset$).

Since $\bar{S}_n$ has empty interior, its complement is a dense open set. That is

$$\overline{X \setminus \bar{S}_n} = X \setminus (\bar{S}_n)^\circ = X$$

Thus if we take any ball $B_d(x, r)$ in X, there is a point $y \in (X \setminus \bar{S}_n) \cap B_d(x, r)$ and then because $X \setminus \bar{S}_n$ is open there is a (smaller) $\delta > 0$ with $B_d(y, \delta) \subset (X \setminus \bar{S}_n) \cap B_d(x, r)$. In fact, making $\delta > 0$ smaller again, there is $\delta > 0$ with $\bar{B}(y, \delta) \subset (X \setminus \bar{S}_n) \cap B_d(x, r)$.

Start with $x_0 \in X$ any point and $r_0 = 1$. Then, by the above reasoning there is a ball

$$\bar{B}_d(x_1, r_1) = \{x \in X : d(x, x_1) \leq r_1\} \subset (X \setminus \bar{S}_1) \cap B_d(x_0, r_0)$$

and $r_1 < r_0/2 \leq 1/2$. We can then find x_2 and $r_2 \leq r_1/2 < 1/2^2$ so that

$$\bar{B}_d(x_2, r_2) \subset (X \setminus \bar{S}_2) \cap B_d(x_1, r_1)$$

and we can continue this process to select $x_1, x_2, \ldots$ and $r_1, r_2, \ldots$ with

$$0 < r_n \leq r_{n-1}/2 < \frac{1}{2^n}, \quad \bar{B}_d(x_n, r_n) \subset (X \setminus \bar{S}_n) \cap B_d(x_{n-1}, r_{n-1}) \quad (n = 1, 2, \ldots)$$

We claim the sequence $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X. This is because $m \geq n \Rightarrow x_m \in B_d(x_n, r_n) \Rightarrow d(x_m, x_n) < r_n < 1/2^n$. So, if n, m are both large

$$d(x_m, x_n) < \min\left(\frac{1}{2^n}, \frac{1}{2^m}\right)$$

is small.

By completeness, $x_\infty = \lim_{n\to\infty} x_n$ exists in X. Since the closed ball $\bar{B}_d(x_n, r_n)$ is a closed set in X and contains all x_m for $m \geq n$, it follows that $x \in \bar{B}_d(x_n, r_n)$ for each n. But $\bar{B}_d(x_n, r_n) \subset X \setminus \bar{S}_n$ and so $x \notin \bar{S}_n$. This is true for all n and so we have the contradiction

$$x \notin \bigcup_{n=1}^\infty \bar{S}_n = X$$

Thus X cannot be a union of a sequence of nowhere dense subsets. □

1.6.4 Corollary. *Let (X, d) be a compact metric space. Then the whole space $S = X$ is of second category in itself.*

Proof. Compact metric spaces are complete. So this follows from the theorem. □

1.7 Banach spaces

1.7.1 Definition. A normed space $(E, \|\cdot\|)$ (over $\mathbb{K}$) is called a *Banach space* (over $\mathbb{K}$) if E is complete in the metric arising from the norm.

1.7.2 Examples. (i) $\mathbb{K}^n$ with the standard Euclidean norm is complete (that is a Banach space).

Proof. Consider a Cauchy sequence $(x_m)_{m=1}^\infty$ in $\mathbb{K}^n$. We write out each term of the sequence as an n-tuple of scalars

$$x_m = (x_{m,1}, x_{m,2}, \ldots, x_{m,n}).$$

Note that, for a fixed j in the range $1 \leq j \leq n$, $|x_{m,j} - x_{p,j}| \leq \|x_m - x_p\|$. It follows that, for fixed j, the sequence of scalars $(x_{m,j})_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$. Thus

$$y_j = \lim_{m\to\infty} x_{m,j}$$

exists in $\mathbb{K}$. Let $y = (y_1, y_2, \ldots, y_n) \in \mathbb{K}^n$. We claim that $\lim_{m\to\infty} x_m = y$, that is we claim $\lim_{m\to\infty} \|x_m - y\|_2 = 0$. But

$$\|x_m - y\|_2 = \sqrt{\sum_{j=1}^n |x_{m,j} - y_j|^2} \to 0$$

as $m \to \infty$. □

(ii) $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space (that is complete, since we already know it is a normed space).

Proof. We will copy the previous proof to a certain extent, but we need some modifications because the last part will be harder.

Consider a Cauchy sequence $(x_m)_{m=1}^\infty$ in ℓ^∞. We write out each term of the sequence as an infinite sequence of scalars

$$x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,j}, \ldots).$$

Note that, for a fixed $j \geq 1$, $|x_{n,j} - x_{m,j}| \leq \|x_n - x_m\|_\infty$. It follows that, for fixed j, the sequence of scalars $(x_{n,j})_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$. Thus

$$y_j = \lim_{n\to\infty} x_{n,j}$$

exists in $\mathbb{K}$. Let $y = (y_1, y_2, \ldots, y_j, \ldots)$. We claim that $\lim_{m\to\infty} x_m = y$ in ℓ^∞, but first we have to know that $y \in \ell^\infty$. Once we know that, what we need to show is that $\lim_{n\to\infty} \|x_n - y\|_\infty = 0$.

To show $y \in \ell^\infty$, we start with the Cauchy condition for $\varepsilon = 1$. It says that there exists $N \geq 1$ so that

$$n, m \geq N \Rightarrow \|x_n - x_m\|_\infty < 1$$

Taking $n = N$ we get

$$m \geq N \Rightarrow \|x_N - x_m\|_\infty < 1$$

Since $|x_{N,j} - x_{m,j}| \leq \|x_N - x_m\|_\infty$ it follows that for each $j \geq 1$ we have

$$m \geq N \Rightarrow |x_{N,j} - x_{m,j}| < 1$$

Letting $m \to \infty$, we find that $|x_{N,j} - y_j| \leq 1$. Thus

$$|y_j| \leq |x_{N,j} - y_j| + |x_{N,j}| \leq 1 + \|x_N\|_\infty$$

holds for $j \geq 1$ and so $\sup_j |y_j| < \infty$. We have verified that $y \in \ell^\infty$.

To show that $\lim_{n\to\infty} \|x_n - y\|_\infty = 0$, we start with $\varepsilon > 0$ and apply the Cauchy criterion to find $N = N_\varepsilon \geq 1$ (not the same N as before) so that

$$n, m \geq N \Rightarrow \|x_n - x_m\|_\infty < \frac{\varepsilon}{2}$$

Hence, for any $j \geq 1$ we have

$$n, m \geq N \Rightarrow |x_{n,j} - x_{m,j}| \leq \|x_n - x_m\|_\infty < \frac{\varepsilon}{2}$$

Fix any $n \geq N$ and let $m \to \infty$ to get

$$|x_{n,j} - y_j| = \lim_{m\to\infty} |x_{n,j} - x_{m,j}| \leq \frac{\varepsilon}{2}$$

So we have

$$n \geq N \Rightarrow \|x_n - y\|_\infty = \sup_{j\geq 1} |x_{n,j} - y_j| \leq \frac{\varepsilon}{2} < \varepsilon$$

This shows $\lim_{n\to\infty} \|x_n - y\|_\infty = 0$, as required. □

(iii) If X is a topological space then $(BC(X), \|\cdot\|_\infty)$ is a Banach space.

(Note that this includes $\ell^\infty = BC(\mathbb{N})$ as a special case. The main difference is that we need to worry about continuity here.)

> Convergence of sequences in the supremum norm corresponds to uniform convergence on X. (See §A.1 for the definition and a few useful facts about uniform convergence.)

Proof. (of the assertion about uniform convergence).

Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $BC(X)$ and $g \in BC(X)$.

First if $f_n \to g$ as $n \to \infty$ (in the metric from the uniform norm on $BC(X)$), we claim that $f_n \to g$ uniformly on X. Given $\varepsilon > 0$ there exists $N \geq 0$ so that

$$n \geq N \Rightarrow d(f_n, g) < \varepsilon \Rightarrow \|f_n - g\| < \varepsilon \Rightarrow \sup_{x \in X} |f_n(x) - g(x)| < \varepsilon$$

From this we see that N satisfies

$$|f_n(x) - g(x)| < \varepsilon \quad \forall x \in X, \forall n \geq N.$$

This means we have established uniform convergence $f_n \to g$ on X.

To prove the converse, assume $f_n \to g$ uniformly on X. Let $\varepsilon > 0$ be given. By uniform convergence we can find $N > 0$ so that

$$n \geq N, x \in X \Rightarrow |f_n(x) - g(x)| < \frac{\varepsilon}{2}.$$

It follows that

$$n \geq N \Rightarrow \sup_{x \in X} |f_n(x) - g(x)| \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so

$$n \geq N \Rightarrow d(f_n, g) = \|f_n - g\| < \varepsilon.$$

Thus $f_n \to g$ in the metric. □

A useful observation is that uniform convergence $f_n \to g$ on X implies pointwise convergence. That is if $f_n \to g$ uniformly, then for each single $x \in X$

$$\lim_{n \to \infty} f_n(x) = g(x)$$

(limit in $\mathbb{K}$ of values at x). Translating that to basics, it means that given one $x \in X$ and $\varepsilon > 0$ there is $N > 0$ so that

$$n \geq N \Rightarrow |f_n(x) - g(x)| < \varepsilon.$$

Uniform convergence means more, that the rate of convergence $f_n(x) \to g(x)$ is 'uniform' (or that, given $\varepsilon > 0$, the same N works for different $x \in X$).

Proof. (that $E = BC(X)$ is complete).

Suppose $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $BC(X)$. We aim to show that the sequence has a limit f in $BC(X)$. We start with the observation that the sequence is 'pointwise Cauchy'. That is if we fix $x_0 \in X$, we have

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x \in X} |f_n(x) - f_m(x)| = \|f_n - f_m\| = d(f_n, f_m)$$

Let $\varepsilon > 0$ be given. We know there is $N > 0$ so that $d(f_n, f_m) < \varepsilon$ holds for all $n, m \geq N$ (because $(f_n)_{n=1}^\infty$ is a Cauchy sequence in the metric d). For the same N we have $|f_n(x_0) - f_m(x_0)| \leq d(f_n, f_m) < \varepsilon \forall n, m \geq N$

Thus $(f_n(x_0))_{n=1}^\infty$ is a Cauchy sequence of scalars (in $\mathbb{K}$). Since $\mathbb{K}$ is complete $\lim_{n\to\infty} f_n(x_0)$ exists in $\mathbb{K}$. This allows us to define $f\colon X \to \mathbb{K}$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in X).$$

We might think we are done now, but all we have now is a pointwise limit of the sequence $(f_n)_{n=1}^\infty$. We need to know more, first that $f \in BC(X)$ and next that the sequence converges to f in the metric d arising from the norm.

We show first that f is bounded on X. From the Cauchy condition (with $\varepsilon = 1$) we know there is $N > 0$ so that $d(f_n, f_m) < 1 \forall n, m \geq N$. In particular if we fix $n = N$ we have

$$d(f_N, f_m) = \sup_{x \in X} |f_N(x) - f_m(x)| < 1 \quad (\forall m \geq N).$$

Now fix $x \in X$ for a moment. We have $|f_N(x) - f_m(x)| < 1$ for all $m \geq N$. Let $m \to \infty$ and we get

$$|f_N(x) - f(x)| \leq 1.$$

This is true for each $x \in X$ and so we have

$$\sup_{x \in X} |f_N(x) - f(x)| \leq 1.$$

We deduce

$$\begin{aligned}\sup_{x\in X}|f(x)| &= \sup_{x\in X}|f_N(x)-f(x)-f_N(x)|\\ &\le \sup_{x\in X}|f_N(x)-f(x)|+|-f_N(x)|\\ &\le 1+\|f_N\|<\infty.\end{aligned}$$

To show that f is continuous, we show that $f_n \to f$ uniformly on X (and appeal to Proposition A.1.2) and once we know $f \in BC(X)$ we can restate uniform convergence of the sequence $(f_n)_{n=1}^\infty$ to f as convergence in the metric of $BC(X)$.

To show uniform convergence, let $\varepsilon > 0$ be given. From the Cauchy condition we know there is $N > 0$ so that $d(f_n, f_m) < \varepsilon/2 \forall n, m \ge N$. In particular if we fix $n \ge N$ we have

$$d(f_n, f_m) = \sup_{x\in X}|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad (\forall m \ge N).$$

Now fix $x \in X$ for a moment. We have $|f_n(x) - f_m(x)| < \varepsilon/2$ for all $m \ge N$. Let $m \to \infty$ and we get

$$|f_n(x) - f(x)| \le \varepsilon/2.$$

This is true for each $x \in X$ and so we have

$$\sup_{x\in X}|f_n(x) - f(x)| \le \varepsilon/2 < \varepsilon.$$

As this is true for each $n \ge N$, we deduce $f_n \to f$ uniformly on X. As a uniform limit of continuous functions, f must be continuous. We already have f bounded and so $f \in BC(X)$. Finally, we can therefore restate $f_n \to f$ uniformly on X as $\lim_{n\to\infty} d(f_n, f) = 0$, which means that f is the limit of the sequence $(f_n)_{n=1}^\infty$ in the metric of $BC(X)$. □

(iv) If we replace X by a compact Hausdorff space K in the previous example, we see that $C(K), \|\cdot\|_\infty)$ is a Banach space.

1.7.3 Lemma. *If $(E, \|\cdot\|_E)$ is a normed space and $F \subseteq E$ is a vector subspace, then F becomes a normed space if we define $\|\cdot\|_F$ (the norm on F) by restriction*

$$\|x\|_F = \|x\|_E \text{ for } x \in F$$

We call $(F, \|\cdot\|_F)$ a subspace *of $(E, \|\cdot\|_E)$.*

Proof. Easy exercise. □

1.7.4 Proposition. *If $(E, \|\cdot\|_E)$ is a Banach space and $(F, \|\cdot\|_F)$ a (normed) subspace, then F is a Banach space (in the subspace norm) if and only if F is closed in E.*

Proof. The issue is completeness of F. It is a general fact about complete metric spaces that a submetric space is complete if and only if it is closed (Proposition 1.4.6). □

1.7.5 Examples. (i) Let K be a compact Hausdorff space and $x_0 \in K$. Then

$$E = \{f \in C(K) : f(x_0) = 0\}$$

is a closed vector subspace of $C(K)$. Hence E is a Banach space in the supremum norm.

Proof. One way to organise the proof is to introduce the point evaluation map $\delta_{x_0} : C(K) \to \mathbb{K}$ given by

$$\delta_{x_0}(f) = f(x_0)$$

One can check that δ_{x_0} is a linear transformation ($\delta_{x_0}(f+g) = (f+g)(x_0) = f(x_0) + g(x_0) = \delta_{x_0}(f) + \delta_{x_0}(g)$; $\delta_{x_0}(\lambda f) = \lambda f(x_0) = \lambda \delta_{x_0}(f)$). It follows then that

$$E = \ker \delta_{x_0}$$

is a vector subspace of $C(K)$.

We can also verify that δ_{x_0} is continuous. If a sequence $(f_n)_{n=1}^\infty$ converges in $C(K)$ to $f \in C(K)$, we have seen above that means $f_n \to f$ uniformly on K. We have also seen this implies $f_n \to f$ pointwise on K. In particular at the point $x_0 \in K$, $\lim_{n\to\infty} f_n(x_0) = f(x_0)$, which means that $\lim_{n\to\infty} \delta_{x_0}(f_n) = \delta_{x_0}(f)$. As this holds for all convergent sequences in $C(K)$, it shows that δ_{x_0} is continuous.

From this it follows that

$$E = \ker \delta_{x_0} = (\delta_{x_0})^{-1}(\{0\})$$

is closed (the inverse image of a closed set $\{0\} \subseteq \mathbb{K}$ under a continuous function). □

(ii) Let
$$c_0 = \{(x_n)_{n=1}^\infty \in \ell^\infty : \lim_{n\to\infty} x_n = 0\}.$$
We claim that c_0 is a closed subspace of ℓ^∞ and hence is a Banach space in the (restriction of) $\|\cdot\|_\infty$.

Proof. We can describe c_0 as the space of all sequences $(x_n)_{n=1}^\infty$ of scalars with $\lim_{n\to\infty} x_n = 0$ (called null sequence sometimes) because convergent sequences in $\mathbb{K}$ are automatically bounded. So the condition we imposed that $(x_n)_{n=1}^\infty \in \ell^\infty$ is not really needed.

Now it is quite easy to see that c_0 is a vector space (under the usual term-by-term vector space operations). If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$ then $\lim_{n\to\infty}(x_n + y_n) = 0$. This shows that $(x_n)_{n=1}^\infty + (y_n)_{n=1}^\infty \in c_0$ if both sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in c_0$. It is no harder to show that $\lambda(x_n)_{n=1}^\infty \in c_0$ if $\lambda \in \mathbb{K}$, $(x_n)_{n=1}^\infty \in c_0$.

To show directly that c_0 is closed in ℓ^∞ is a bit tricky because elements of c_0 are themselves sequences of scalars and to show $c_0 \subseteq \ell^\infty$ is closed we show that whenever a sequence $(z_n)_{n=1}^\infty$ of terms $z_n \in c_0$ converges in ℓ^∞ to a limit $w \in \ell^\infty$, then $w \in c_0$.

To organise what we have to do we can write out each $z_n \in c_0$ as a sequence of scalars by using a double subscript
$$z_n = (z_{n,1}, z_{n,2}, z_{n,3}, \ldots) = (z_{n,j})_{j=1}^\infty$$
(where the $z_{n,j} \in \mathbb{K}$ are scalars). We can write $w = (w_j)_{j=1}^\infty$ and now what we are assuming is that $z_n \to w$ in $(\ell^\infty, \|\cdot\|_\infty)$. That means
$$\lim_{n\to\infty} \|z_n - w\|_\infty = \lim_{n\to\infty} \left(\sup_{j\geq 1} |z_{n,j} - w_j| \right) = 0.$$
To show $w \in c_0$, start with $\varepsilon > 0$ given. Then we can find $N \geq 0$ so that $\|z_n - w\|_\infty < \varepsilon/2$ holds for all $n \geq N$. In particular $\|z_N - w\|_\infty < \varepsilon/2$. As $z_N \in c_0$ we know $\lim_{j\to\infty} z_{Nj} = 0$. Thus there is $j_0 > 0$ so that $|z_{N,j}| < \varepsilon/2$ holds for all $j \geq j_0$. For $j \geq j_0$ we have then
$$|w_j| \leq |w_j - z_{N,j}| + |z_{N,j}| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$
This shows $\lim_{j\to\infty} w_j = 0$ and $w \in c_0$.

This establishes that c_0 is closed in ℓ^∞ and completes the proof that c_0 is a Banach space. $\square$

(iii) There is another approach to showing that c_0 is a Banach space.

Let $\mathbb{N}^*$ be the one-point compactification of $\mathbb{N}$ with one extra point (at 'infinity') added on. We will write ∞ for this extra point. Each sequence $(x_n)_{n=1}^\infty$ defines a function $f \in C(\mathbb{N}^*)$ via $f(n) = x_n$ for $n \in \mathbb{N}$ and $f(\infty) = 0$.

In fact one may identify $C(\mathbb{N}^*)$ with the sequence space

$$c = \{(x_n)_{n=1}^\infty : x_n \in \mathbb{K} \forall n \text{ and } \lim_{n\to\infty} x_n \text{ exists in } \mathbb{K}\}.$$

So c is the space of all convergent sequences (also contained in ℓ^∞) and the identification is that the sequence $(x_n)_{n=1}^\infty$ corresponds to the function $f \in C(\mathbb{N}^*)$ via $f(n) = x_n$ for $n \in \mathbb{N}$ and $f(\infty) = \lim_{n\to\infty} x_n$. The supremum norm (on $C(\mathbb{N}^*)$) is $\|f\|_\infty = \sup_{x\in\mathbb{N}^*} |f(x)| = \sup_{n\in\mathbb{N}} |f(n)| = \|(x_n)_{n=1}^\infty\|_\infty$ (because $\mathbb{N}$ is dense in $\mathbb{N}^*$).

In this way we can see that c_0 corresponds to the space of functions in $C(\mathbb{N}^*)$ that vanish at ∞. Using the first example, we see again that c_0 is a Banach space.

Being a subspace of ℓ^∞ it must be closed in ℓ^∞ by Proposition 1.7.4.

1.7.6 Remark. We next give a criterion in terms of series that is sometimes useful to show that a normed space is complete.

Because a normed space has both a vector space structure (and so addition is possible) and a metric (means that convergence makes sense) we can talk about infinite series converging in a normed space.

1.7.7 Definition. If $(E, \|\cdot\|)$ is a normed space then a *series* in E is just a sequence $(x_n)_{n=1}^\infty$ of terms $x_n \in E$.

We define the *partial sums* of the series to be

$$s_n = \sum_{j=1}^n x_j.$$

We say that the *series converges* in E if the sequence of partial sums has a limit — $\lim_{n\to\infty} s_n$ exists in E, or there exists $s \in E$ so that

$$\lim_{n\to\infty} \left\| \left(\sum_{j=1}^n x_j \right) - s \right\| = 0$$

We write $\sum_{n=1}^\infty x_n$ when we mean to describe a series and we also write $\sum_{n=1}^\infty x_n$ to stand for the value s above in case the series does converge. As for scalar series, we may write that $\sum_{n=1}^\infty x_n$ 'does not converge' if the sequence of partial sums has no limit in E.

We say that a series $\sum_{n=1}^\infty x_n$ is *absolutely convergent* if $\sum_{n=1}^\infty \|x_n\| < \infty$. (Note that $\sum_{n=1}^\infty \|x_n\|$ is a real series of positive terms and so has a monotone increasing sequence of partial sums. Therefore the sequence of its partial sums either converges in $\mathbb{R}$ or increases to ∞.)

1.7.8 Proposition. *Let $(E, \|\cdot\|)$ be a normed space. Then E is a Banach space (that is complete) if and only if each absolutely convergent series $\sum_{n=1}^\infty x_n$ of terms $x_n \in E$ is convergent in E.*

Proof. Assume E is complete and $\sum_{n=1}^\infty \|x_n\| < \infty$. Then the parial sums of this series of positive terms

$$S_n = \sum_{j=1}^n \|x_j\|$$

must satisfy the Cauchy criterion. That is for $\varepsilon > 0$ given there is N so that $|S_n - S_m| < \varepsilon$ holds for all $n, m \geq N$. If we take $n > m \geq N$, then

$$|S_n - S_m| = \left| \sum_{j=1}^n \|x_j\| - \sum_{j=1}^m \|x_j\| \right| = \sum_{j=m+1}^n \|x_j\| < \varepsilon.$$

Then if we consider the partial sums $s_n = \sum_{j=1}^n x_j$ of the series $\sum_{n=1}^\infty x_n$ we see that for $n > m \geq N$ (same N)

$$\|s_n - s_m\| = \left\| \sum_{j=1}^n x_j - \sum_{j=1}^m x_j \right\| = \left\| \sum_{j=m+1}^n x_j \right\| \leq \sum_{j=m+1}^n \|x_j\| < \varepsilon.$$

It follows from this that the sequence $(s_n)_{n=1}^\infty$ is Cauchy in E. As E is complete, $\lim_{n\to\infty} s_n$ exists in E and so $\sum_{n=1}^\infty x_n$ converges.

For the converse, assume that all absolutely convergent series in E are convergent. Let $(u_n)_{n=1}^\infty$ be a Cauchy sequence in E. Using the Cauchy condition with $\varepsilon = 1/2$ we can find $n_1 > 0$ so that

$$n, m \geq n_1 \Rightarrow \|u_n - u_m\| < \frac{1}{2}.$$

Next we can (using the Cauchy condition with $\varepsilon = 1/2^2$) find $n_2 > 1$ so that

$$n, m \geq n_2 \Rightarrow \|u_n - u_m\| < \frac{1}{2^2}.$$

We can further assume (by increasing n_2 if necessary) that $n_2 > n_1$. Continuing in this way we can find $n_1 < n_2 < n_3 < \cdots$ so that

$$n, m \geq n_j \Rightarrow \|u_n - u_m\| < \frac{1}{2^j}.$$

Consider now the series $\sum_{j=1}^{\infty} x_j = \sum_{j=1}^{\infty}(u_{n_{j+1}} - u_{n_j})$. It is absolutely convergent because

$$\sum_{j=1}^{\infty} \|x_j\| = \sum_{j=1}^{\infty} \|u_{n_{j+1}} - u_{n_j}\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$

By our assumption, it is convergent. Thus its sequence of partial sums

$$s_J = \sum_{j=1}^{J}(u_{n_{j+1}} - u_{n_j}) = u_{n_{J+1}} - u_{n_1}$$

has a limit in E (as $J \to \infty$). It follows that

$$\lim_{J\to\infty} u_{n_{J+1}} = u_{n_1} + \lim_{J\to\infty}(u_{n_{J+1}} - u_{n_1})$$

exists in E. So the Cauchy sequence $(u_n)_{n=1}^{\infty}$ has a convergent subsequence. By Lemma 1.4.8 E is complete. □

1.7.9 Definition. For $1 \leq p < \infty$, ℓ^p denotes the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ which satisfy

$$\sum_{n=1}^{\infty} |a_n|^p < \infty.$$

1.7.10 Proposition. *ℓ^p is a vector space (under the usual term-by-term addition and scalar multiplication for sequences). It is a Banach space in the norm*

$$\|(a_n)_n\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$$

The proof will require the following three lemmas.

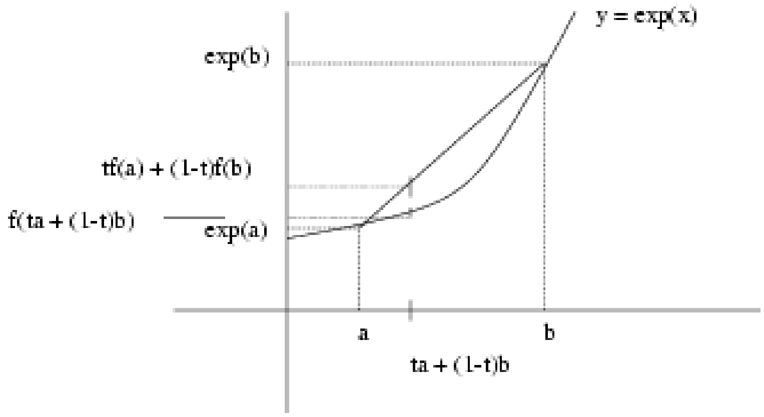
1.7.11 Lemma. *Suppose $1 < p < \infty$ and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{for } a, b \geq 0.$$

Proof. If either $a = 0$ or $b = 0$, then the inequality is clearly true.

The function $f(x) = e^x$ is a convex function of x. This means that $f''(x) \geq 0$, or geometrically that

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t)f(\beta) \text{ for } 0 \leq t \leq 1.$$



Put $t = \frac{1}{p}$ and $1 - t = \frac{1}{q}$ to get

$$e^{\frac{\alpha}{p} + \frac{\beta}{q}} \leq \frac{1}{p}e^{\alpha} + \frac{1}{q}e^{\beta}$$

or

$$(e^{\alpha/p})(e^{\beta/q}) \leq \frac{1}{p}e^{\alpha} + \frac{1}{q}e^{\beta}.$$

Put $a = e^{\alpha/p}$ and $b = e^{\beta/q}$ (or perhaps $\alpha = p \log a$, $\beta = q \log b$) to get the result. □

1.7.12 Lemma (Hölder's inequality)**.** *Suppose* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *(if* $p = 1$ *this is interpreted to mean* $q = \infty$ *and values of* p *and* q *satisfying this relationship are called* **conjugate exponents***). For* $(a_n)_n \in \ell^p$ *and* $(b_n)_n \in \ell^q$*,*

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \|(a_n)_n\|_p \, \|(b_n)_n\|_q.$$

(This means both that the series on the left converges and that the inequality is true.)

Proof. This inequality is quite elementary if $p = 1$ and $q = \infty$. Suppose $p > 1$. Let

$$
\begin{aligned}
A &= \|(a_n)_n\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \\
B &= \|(b_n)_n\|_q = \left(\sum_{n=1}^{\infty} |b_n|^q\right)^{1/q}
\end{aligned}
$$

If either $A = 0$ or $B = 0$, the inequality is trivially true. Otherwise, use Lemma 1.7.11 with $a = |a_n|/A$ and $b = |b_n|/B$ to get

$$
\begin{aligned}
\frac{|a_n b_n|}{AB} &\leq \frac{1}{p}\frac{|a_n|^p}{A^p} + \frac{1}{q}\frac{|b_n|^q}{B^q} \\
\sum_{n=1}^{\infty} \frac{|a_n b_n|}{AB} &\leq \frac{1}{p}\sum \frac{|a_n|^p}{A^p} + \frac{1}{q}\sum \frac{|b_n|^q}{B^q} \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}
$$

Hence

$$\sum |a_n b_n| \leq AB$$

□

1.7.13 Remark. For $p = 2$ and $q = 2$, Hölder's inequality reduces to an infinite-dimensional version of the Cauchy-Schwarz inequality

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_n |a_n|^2\right)^{1/2} \left(\sum_n |y_n|^2\right)^{1/2}.$$

1.7.14 Lemma (Minkowski's inequality)**.** *If $x = (x_n)_n$ and $y = (y_n)_n$ are in ℓ^p ($1 \leq p \leq \infty$) then so is $(x_n + y_n)_n$ and*

$$\|(x_n + y_n)_n\|_p \leq \|(x_n)_n\|_p + \|(y_n)_n\|_p$$

Proof. This is quite trivial to prove for $p = 1$ and for $p = \infty$ (we have already encountered the case $p = \infty$). So suppose $1 < p < \infty$.

First note that

$$\begin{aligned}
|x_n + y_n| &\leq |x_n| + |y_n| \leq 2\max(|x_n|, |y_n|) \\
|x_n + y_n|^p &\leq 2^p \max(|x_n|^p, |y_n|^p) \\
&\leq 2^p\Big(|x_n|^p + |y_n|^p\Big) \\
\sum_n |x_n + y_n|^p &\leq 2^p\Big(\sum_n |x_n|^p + \sum_n |y_n|^p\Big)
\end{aligned}$$

This shows that $(x_n + y_n)_n \in \ell^p$.

Next, to show the inequality,

$$\begin{aligned}
\sum_n |x_n + y_n|^p &= \sum_n |x_n + y_n|\,|x_n + y_n|^{p-1} \\
&= \sum_n |x_n|\,|x_n + y_n|^{p-1} + \sum_n |y_n|\,|x_n + y_n|^{p-1}
\end{aligned}$$

Write $\sum_n |x_n|\,|x_n + y_n|^{p-1} = \sum_n a_n b_n$ where $a_n = |x_n|$ and $b_n = |x_n + y_n|^{p-1}$. Then we have $(a_n)_n \in \ell^p$ and $(b_n)_n \in \ell^q$ because

$$\begin{aligned}
\sum_n b_n^q &= \sum_n |x_n + y_n|^{(p-1)q} \\
&= \sum_n |x_n + y_n|^p < \infty
\end{aligned}$$

where we have used the relation $\frac{1}{p} + \frac{1}{q} = 1$ to show $(p-1)q = p$. From Lemma 1.7.12 we deduce

$$\begin{aligned}
\sum_n |x_n||x_n + y_n|^{p-1} &\leq \left(\sum_n |x_n|^p\right)^{1/p}\left(\sum_n |x_n + y_n|^{(p-1)q}\right)^{1/q} \\
&= \|(x_n)_n\|_p \|(x_n + y_n)_n\|_p^{p/q}
\end{aligned}$$

Similarly

$$\sum_n |x_n||x_n + y_n|^{p-1} \leq \|(y_n)_n\|_p \|(x_n + y_n)_n\|_p^{p/q}$$

Adding the two inequalities, we get

$$\|(x_n + y_n)_n\|_p^p \leq \|(x_n)_n\|_p \|(x_n + y_n)_n\|_p^{p/q} + \|(y_n)_n\|_p \|(x_n + y_n)_n\|_p^{p/q}.$$

Now, if $\|(x_n + y_n)_n\|_p = 0$ then the inequality to be proved is clearly satisfied. If $\|(x_n + y_n)_n\|_p \neq 0$, we can divide across by $\|(x_n + y_n)_n\|_p^{p/q}$ to obtain

$$\|(x_n + y_n)_n\|_p^{p-p/q} \leq \|(x_n)_n\|_p + \|(y_n)_n\|_p.$$

Since $p - \frac{p}{q} = 1$, this is the desired inequality. □

Proof. (of Proposition 1.7.10): It follows easily from Lemma 1.7.14 that ℓ^p is a vector space and that $\|\cdot\|_p$ is a norm on it (in fact Minkowski's inequality is just the triangle inequality for the ℓ^p-norm).

To show that ℓ^p is complete, we show that every absolutely convergent series $\sum_k x_k$ in ℓ^p is convergent. (That is we use Proposition 1.7.8.)

Write $x_k = (x_{k,n})_n = (x_{k,1}, x_{k,2}, \ldots)$ for each k. Notice that

$$|x_{k,n}| \leq \|x_k\|_p = \left(\sum_n |x_{k,n}|^p \right)^{1/p}$$

Therefore $\sum_k |x_{k,n}| \leq \sum_k \|x_k\|_p < \infty$ for each k and it makes sense to write

$$y_n = \sum_k x_{k,n}$$

(and $y_n \in \mathbb{K}$).

Now, for any $N \geq 1$,

$$\begin{aligned}
&\left(\sum_{n=1}^N |y_n|^p \right)^{1/p} \\
&= \lim_{K \to \infty} \left(\sum_{n=1}^N \left| \sum_{k=1}^K x_{k,n} \right|^p \right)^{1/p} \\
&= \lim_{K \to \infty} \| \quad (x_{1,1}, \; x_{1,2}, \; \ldots, \; x_{1,N}, \; 0, \; 0, \; \ldots) \\
&\qquad\qquad + (x_{2,1}, \; x_{2,2}, \; \ldots, \; x_{2,N}, \; 0, \; 0, \; \ldots) \\
&\qquad\qquad \vdots \\
&\qquad\qquad + (x_{K,1}, \; x_{K,2}, \; \ldots, \; x_{K,N}, \; 0, \; 0, \; \ldots) \; \|_p \\
&\leq \lim_{K \to \infty} \sum_{k=1}^K \|(x_{k,1}, x_{k,2}, \ldots, x_{k,N}, 0, 0, \ldots)\|_p
\end{aligned}$$

(using Minkowski's inequality)

$$\begin{aligned} &\leq \lim_{K\to\infty} \sum_{k=1}^{K} \|x_k\|_p \\ &= \sum_{k=1}^{\infty} \|x_k\|_p < \infty \end{aligned}$$

Letting $N \to \infty$, this shows that $y = (y_n)_n \in \ell^p$.

Applying similar reasoning to $y - \sum_{k=1}^{K_0} x_k$ (for any given $K_0 \geq 0$) shows that

$$\begin{aligned} \left\| y - \sum_{k=1}^{K_0} x_k \right\|_p &\leq \sum_{K_0+1}^{\infty} \|x_k\|_p \\ &\to 0 \qquad \text{as } K_0 \to \infty. \end{aligned}$$

In other words the series $\sum_k x_k$ converges to y in ℓ^p. □

1.7.15 Examples. (i) We define, for $1 \leq p < \infty$,

$$\mathcal{L}^p([0,1]) = \{f : [0,1] \to \mathbb{K} : f \text{ measurable and } \int_0^1 |f(x)|^p \, dx < \infty\}.$$

On this space we define

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

The idea is that we have replaced sums used in ℓ^p by integrals over the unit interval $[0,1]$. It is perhaps natural to then allow measurable functions as these are the right class to consider in the context of integration. One might be tempted to be more restrictive and (say) only allow continuous f but this causes problems we will mention later.

We can use the same ideas exactly as we used in the proofs of Hölder's inequality (Lemma 1.7.12) and Minkowski's inequality (Lemma 1.7.14) to show integral versions of them. We end up showing that

$$f, g \in \mathcal{L}^p([0,1]) \Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

(the triangle inequality). It is not at all hard to see that $\|\lambda f\|_p = |\lambda| \|f\|_p$ and we are well on the way to showing $\|\cdot\|_p$ is a norm on $\mathcal{L}^p([0,1])$. However, it is not a norm. It is only a seminorm because $\|f\|_p = 0$ implies only that

$\{x \in [0,1] : f(x) \neq 0\}$ has measure 0 (total length 0). When something is true except for a set of total length 0 we say it is true *almost everywhere* [with respect to length measure or Lebesgue measure on $\mathbb{R}$].

There is a standard way to get from a seminormed space to a normed space, by taking equivalence classes. We define an equivalence relation on $\mathcal{L}^p([0,1])$ by $f \sim g$ if $\|f-g\|_p = 0$ (which translates in this case to $f(x) = g(x)$ almost everywhere). We can then turn the set of equivalence classes

$$L^p([0,1]) = \{[f] : f \in \mathcal{L}^p([0,1])\}$$

into a vector space by defining

$$[f] + [g] = [f+g], \quad \lambda[f] = [\lambda f].$$

There is quite a bit of checking to do to show this is well-defined. Anytime we define operations on equivalence classes in terms of representatives of the equivalence classes, we have to show that the operation is independent of the choice of representatives.

Next we can define a norm on $L^p([0,1])$ by $\|[f]\|_p = \|f\|_p$ (and again we have to show this is well defined and actually leads to a norm). Finally we end up with a normed space $(L^p([0,1]), \|\cdot\|_p)$ and it is in fact a Banach space. The proof of that needs some facts from measure theory and can be based on Proposition 1.7.8.

If $\sum_{n=1}^\infty \|f_n\|_p < \infty$, let

$$g(x) = \lim_{N\to\infty} \sum_{n=1}^N |f_n(x)| \quad (x \in [0,1])$$

with the understanding that $g(x) \in [0, +\infty]$. By the monotone convergence theorem

$$\int_0^1 g(x)^p \, dx = \lim_{N\to\infty} \int_0^1 \left(\sum_{n=1}^N |f_n(x)| \right)^p dx$$

From Minkowski's inequality, we get

$$\begin{aligned} \left(\int_0^1 \left(\sum_{n=1}^N |f_n(x)| \right)^p dx \right)^{1/p} &= \left\| \sum_{n=1}^N |f_n| \right\|_p \leq \sum_{n=1}^N \| |f_n| \|_p \\ &= \sum_{n=1}^N \|f_n\|_p \leq \sum_{n=1}^\infty \|f_n\|_p \end{aligned}$$

and it follows then that

$$\int_0^1 g(x)^p\,dx \le \left(\sum_{n=1}^\infty \|f_n\|_p\right)^p < \infty$$

and so $g(x) < \infty$ for almost every $x \in [0,1]$.

On the set where $g(x) < \infty$, we can define

$$f(x) = \sum_{n=1}^\infty f_n(x) = \lim_{N\to\infty} \sum_{n=1}^N f_n(x)$$

(and on the set of measure zero where $g(x) = \infty$ we can define $f(x) = 0$). Then f is measurable. From $|f(x)| \le g(x)$ and the above, $f \in \mathcal{L}^p([0,1])$. To show $\lim_{N\to\infty} \left\| \left(\sum_{n=1}^N f_n\right) - f \right\|_p = 0$, use the fact that

$$\left| \left(\sum_{n=1}^N f_n(x)\right) - f(x) \right| = \left| \sum_{n=N+1}^\infty f_n(x) \right| \le g(x)$$

(for almost every $x \in [0,1]$). Since we know $\int_0^1 g(x)^p\,dx < \infty$, the Lebesgue dominated convergence theorem allows us to conclude

$$\begin{aligned} \lim_{N\to\infty} \left\| \left(\sum_{n=1}^N f_n\right) - f \right\|_p^p &= \lim_{N\to\infty} \int_0^1 \left| \left(\sum_{n=1}^N f_n(x)\right) - f(x) \right|^p dx \\ &= \int_0^1 \left| \lim_{N\to\infty} \left(\sum_{n=1}^N f_n(x)\right) - f(x) \right|^p dx = 0 \end{aligned}$$

Thus we have proved $\sum_{n=1}^\infty f_n$ converges to f in $\mathcal{L}^p([0,1])$.

We can never quite forget that $L^p([0,1])$ is not actually a space of (measurable) functions but really a space of (almost everywhere) equivalence classes of functions. However, it is usual not to dwell on this point. What it does mean is that if you find yourself discussing $f(1/2)$ or any specific single value of $f \in L^p([0,1])$, you are doing something wrong. The reason is that as $f \in L^p([0,1])$ is actually an equivalence class it is then possible to change the value $f(1/2)$ arbitrarily without changing the element of $L^p([0,1])$ we are considering.

(ii) There are further variations on $L^p([0,1])$ which are useful in different contexts. We could replace $[0,1]$ by another interval $[a,b]$ ($a < b$) and replace $\int_0^1$ by $\int_a^b$. For what we have discussed so far, everything will go through as before. We get $(L^p([a,b]), \|\cdot\|_p)$.

(iii) We can also define $L^p(\mathbb{R})$ where $\mathcal{L}^p(\mathbb{R})$ consists of measurable functions $f\colon \mathbb{R} \to \mathbb{K}$ with $\int_{-\infty}^{\infty} |f(x)|^p\,dx < \infty$. Then we take (as before) almost-everywhere equivalence classes of $f \in \mathcal{L}^p(\mathbb{R})$ to be $L^p(\mathbb{R})$ and we take the norm $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p\,dx\right)^{1/p}$. Again we get a Banach space $(L^p(\mathbb{R}), \|\cdot\|_p)$ for $1 \leq p < \infty$.

(iv) In fact we can define L^p in a more general context that includes all the examples ℓ^p, $L^p([a,b])$ and $L^p(\mathbb{R})$ as special cases. Let (X, Σ, μ) be a measure space. This means X is a set, Σ is a collection of subsets of X with certain properties, and μ is a function that assigns a measure (or mass or length or volume) in the range $[0,\infty]$ to each set in Σ. More precisely Σ should be a σ-algebra of subsets of X — contains the empty set and X itself, closed under taking complements and countable unions. And $\mu\colon \Sigma \to [0,\infty]$ should have $\mu(\emptyset) = 0$ and be countably additive (which means that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ if $E_1, E_2, \ldots \in \Sigma$ are disjoint). Then

$$\mathcal{L}^p(X,\Sigma,\mu) = \left\{ f\colon X \to \mathbb{K} : f \text{ measurable}, \int_X |f(x)|^p\,d\mu(x) < \infty \right\},$$

$L^p(X,\Sigma,\mu)$ consists of equivalence classes of elements in $\mathcal{L}^p(X,\Sigma,\mu)$ where $f \sim g$ means that $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$. We say that $f = g$ *almost everywhere with respect to* μ if $f \sim g$ (and sometimes write $f = g$ a.e. $[\mu]$). On $L^p(X,\Sigma,\mu)$ we take the norm

$$\|f\|_p = \left(\int_X |f(x)|^p\,d\mu(x) \right)^{1/p}.$$

Then $(L^p(X,\Sigma,\mu), \|\cdot\|_p)$ is a Banach space ($1 \leq p < \infty$).

To see why the examples $L^p([0,1])$, $L^p([a,b])$ and $L^p(\mathbb{R})$ are special cases of $L^p(X,\Sigma,\mu)$ we take μ to be Lebesgue (length) measure on the line. The need for Σ is then significant — we cannot assign a length to every subset of $\mathbb{R}$ and keep the countable additivity property. So Σ has to be Lebesgue-measurable subsets of $[0,1]$, $[a,b]$ or $\mathbb{R}$ (or Borel measurable subsets). There is one reason why $L^p([0,1])$ is a little different from the rest. In that case we are dealing with a probability space (X, Σ, μ), meaning a measure space where $\mu(X) = 1$.

To see why ℓ^p is also an $L^p(X, \Sigma, \mu)$, we take $X = \mathbb{N}$, Σ to be all subsets of $\mathbb{N}$ and μ to be counting measure. This means that for $E \subset \mathbb{N}$ finite $\mu(E)$ is the number of elements in E and for E infinite, $\mu(E) = \infty$. In this case we can also think of functions $f : \mathbb{N} \to \mathbb{K}$ as sequences $(f(n))_{n=1}^{\infty}$ of scalars and $\int_X |f(n)|^p \, d\mu(n) = \sum_{n=1}^{\infty} |f(n)|^p$ for counting measure μ. Moreover, the only set of measure 0 for counting measure is the empty set. Thus there is no need to take almost everywhere equivalence classes when dealing with this special case.

(v) We have avoided dealing with L^∞ so far, because the formulae are slightly different. Looking back to the comparison between ℓ^∞ and ℓ^p, we want to replace the condition on the integral of $|f|^p$ being finite by a supremum. We might like to describe $\mathcal{L}^\infty(X, \Sigma, \mu)$ as measurable $f : X \to \mathbb{K}$ with

$$\sup_{x \in X} |f(x)| < \infty,$$

but in keeping with the case of $\mathcal{L}^p$ we also want to take a.e $[\mu]$ equivalence classes of such f. The problem is that while changing a function on a set of measure 0 does not change its integral, it can change its supremum. Hence we need a variation of the supremum that ignores sets of measure 0. This is known as the essential supremum and it can be defined as

$$\text{ess-sup}(f) = \inf \left\{ \sup_{x \in X \setminus E} |f(x)| : E \subset X, \mu(E) = 0 \right\}$$

or

$$\text{ess-sup}(f) = \inf \left\{ \sup_{x \in X} |g(x)| : g \sim f \right\}$$

(using $g \sim f$ to mean that g is a measurable function equal to f almost everywhere). We then define

$$\mathcal{L}^\infty(X, \Sigma, \mu) = \{ f : X \to \mathbb{K} : f \text{ measurable, ess-sup}(f) < \infty \}$$

and $L^\infty(X, \Sigma, \mu)$ to be the almost everywhere equivalence classes of $f \in \mathcal{L}^\infty(X, \Sigma, \mu)$. With the norm

$$\|f\|_\infty = \text{ess-sup}(f)$$

we get a Banach space $(L^\infty(X,\Sigma,\mu),\|\cdot\|_\infty)$.

In the case (X,Σ,μ) is $X=\mathbb{N}$ with counting measure, we can verify that L^∞ is just ℓ^∞ again. For $X=[0,1]$ we get $L^\infty([0,1])$ (using $\mu=$ Lebesgue measure) and we also have $L^\infty([a,b])$ and $L^\infty(\mathbb{R})$.

Of course there are many details omitted here, to verify that everything is as claimed, that the norms are well defined, that the spaces are complete, and so on.

(vi) We can further consider L^p ($1\le p\le\infty$) in other cases, like $L^p(\mathbb{R}^n)$ where we take n-dimensional Lebesgue measure for μ (on $X=\mathbb{R}^n$).

1.8 Linear operators

1.8.1 Theorem. *Let $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ be normed spaces and let $T\colon E\to F$ be a linear transformation. Then the following are equivalent statements about T.*

(i) T is continuous.

(ii) T is continuous at $0\in E$.

(iii) There exists $M\ge 0$ so that $\|Tx\|_F\le M\|x\|_E$ holds for all $x\in E$.

(iv) T is a Lipschitz mapping, that is there exists $M\ge 0$ so that $\|Tx-Ty\|_F\le M\|x-y\|_E$ holds for all $x,y\in E$.

(v) T is uniformly continuous.

Proof. Our strategy for the proof is to show (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i).

(i) ⇒ (ii) Obvious

(ii) ⇒ (iii) By continuity at 0 (with $\varepsilon=1$), there is $\delta>0$ so that

$$\|x\|_E=\|x-0\|_E<\delta\Rightarrow\|Tx-T0\|_F=\|Tx\|_F<1$$

Then for any $y\in E$ with $y\neq 0$ we can take for example $x=(\delta/2)y/\|y\|_E$ to get $\|x\|=\delta/2<\delta$ and so conclude

$$\|Tx\|_F=\left\|T\left(\frac{\delta}{2\|y\|_E}y\right)\right\|_F=\left\|\frac{\delta}{2\|y\|_E}T(y)\right\|_F=\frac{\delta}{2\|y\|_E}\|Ty\|_F<1.$$

Thus $\|Ty\|_F\le(2/\delta)\|y\|_E$ for all $y\in E$ apart from $y=0$. But for $y=0$ this inequality is also true and so we get (iii) with $M=2/\delta>0$.

(iii) ⇒ (iv) We have

$$\|Tx - Ty\|_F = \|T(x-y)\|_F \leq M\|x-y\|_E$$

by linearity of T and (iii).

(iv) ⇒ (v) Given $\varepsilon > 0$, take $\delta = \varepsilon/(M+1) > 0$, then

$$\|x-y\|_E < \delta \Rightarrow \|Tx - Ty\|_F \leq M\|x-y\|_E < M\delta = \frac{\varepsilon M}{M+1} < \varepsilon.$$

Thus we have uniform continuity of T (see 1.4.10).

(v) ⇒ (i) Obvious (by Proposition 1.4.11).

□

1.8.2 Definition. We usually refer to a linear transformation $T \colon E \to F$ between normed spaces that satisfies the condition (iii) of Theorem 1.8.1 above as a *bounded linear operator* (or sometimes just as a *linear operator*).

1.8.3 Remark. From Theorem 1.8.1 we see that bounded is the same as continuous for a linear operator between normed spaces. There are very few occasions in functional analysis when we want to consider linear transformation that fail to be continuous. At least that is so in the elementary theory. When we encounter discontinuous linear transformations in this course it will be in the context of unpleasant phenomena or counterexamples.

1.8.4 Definition. If $T \colon E \to F$ is a bounded linear operator between normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, then we define the operator norm of T to be

$$\|T\|_{\text{op}} = \inf\{M \geq 0 : \|Tx\|_F \leq M\|x\|_E \forall x \in E\}$$

1.8.5 Proposition. *If $T \colon E \to F$ is a bounded linear operator between normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, then*

(a) $\|T\|_{\text{op}} = \inf\{M \geq 0 : \|Tx - Ty\|_F \leq M\|x-y\|_E \forall x, y \in E\}$ *(thus $\|T\|_{\text{op}}$ is the smallest possible Lipschitz constant for T);*

(b) $\|T\|_{\text{op}} = \sup\{\|Tx\|_F : x \in E, \|x\|_E = 1\}$ *(provided $E \neq \{0\}$ or if we interpret the right hand side as 0 in case we have the supremum of the empty set);*

(c) $\|T\|_{\mathrm{op}} = \sup\{\|Tx\|_F : x \in E, \|x\|_E \leq 1\}$*;*

(d) $\|T\|_{\mathrm{op}} = \sup\left\{\frac{\|Tx\|_F}{\|x\|_E} : x \in E, x \neq 0\right\}$ *(again provided* $E \neq \{0\}$ *or if we interpret the right hand side as* 0 *in case we have the supremum of the empty set).*

Proof. Exercise. □

1.8.6 Examples. (i) We claim that if $1 \leq p_1 < p_2 \leq \infty$, then $\ell^{p_1} \subseteq \ell^{p_2}$ and the inclusion operator

$$\begin{aligned} T \colon \ell^{p_1} &\to \ell^{p_2} \\ Tx &= x \end{aligned}$$

has $\|T\|_{op} = 1$.

Notice that this is nearly obvious for $\ell^1 \subseteq \ell^\infty$ as $\sum_{n=1}^\infty |x_n| < \infty$ implies $\lim_{n\to\infty} x_n = 0$ and so the sequence $(x_n)_{n=1}^\infty \in c_0 \subseteq \ell^\infty$.

In fact we can dispense with the case $p_2 = \infty$ first because it is a little different from the other cases. If $(x_n)_{n=1}^\infty \in \ell^{p_1}$, then $\sum_{n=1}^\infty |x_n|^p < \infty$ and so $\lim_{n\to\infty} x_n = 0$. Thus, again, we have $(x_n)_{n=1}^\infty \in c_0 \subseteq \ell^\infty$. We also see that for any fixed m, $|x_m|^p \leq \sum_{n=1}^\infty |x_n|^p = \|(x_n)_{n=1}^\infty\|_{p_1}^{p_1}$ and so $|x_m| \leq \|x\|_{p_1}$ for $x = (x_n)_{n=1}^\infty$. Thus

$$\|x\|_\infty = \sup_m |x_m| \leq \|x\|_{p_1}$$

and this means $\|Tx\|_\infty \leq M\|x\|_{p_1}$ with $M = 1$. So $\|T\|_{op} \leq 1$. To show $\|T\|_{op} \geq 1$ consider the sequence $(1, 0, 0, \ldots)$ which has $\|Tx\|_\infty = \|x\|_\infty = 1 = \|x\|_{p_1}$.

Now consider $1 \leq p_1 \leq p_2 < \infty$. If $x = (x_n)_{n=1}^\infty \in \ell^{p_1}$ and $\|x\|_{p_1} \leq 1$, then we have

$$\sum_{n=1}^\infty |x_n|^{p_1} = \|x\|_{p_1}^{p_1} \leq 1$$

and so $|x_n| \leq 1$ for all n. It follows that

$$\sum_{n=1}^\infty |x_n|^{p_2} = \sum_{n=1}^\infty |x_n|^{p_1} |x_n|^{p_2 - p_1} \leq \sum_{n=1}^\infty |x_n|^{p_1} \leq 1$$

and so $x \in \ell^{p_2}$ (if $\|x\|_{p_1} \leq 1$). For the remaining $x \in \ell^{p_1}$ with $\|x\|_{p_1} \geq 1$, we have $y = x/\|x\|_{p_1}$ of norm $\|y\|_{p_1} = 1$. Hence $y \in \ell^{p_2}$ and so $x = \|x\|_{p_1} y \in \ell^{p_2}$. This shows $\ell^{p_1} \subset \ell^{p_2}$.

We saw above that $\|x\|_{p_1} \leq 1 \Rightarrow \|Tx\|_{p_2} = \|x\|_{p_2} \leq 1$ and this tells us $\|T\|_{op} \leq 1$. To show that the norm is not smaller than 1, consider the case $x = (1, 0, 0, \ldots)$ which has $\|Tx\|_{p_2} = \|x\|_{p_2} = 1 = \|x\|_{p_1}$.

(ii) If $1 \leq p_1 < p_2 \leq \infty$, then $L^{p_2}([0,1]) \subseteq L^{p_1}([0,1])$ and the inclusion operator

$$\begin{aligned} T \colon L^{p_2}([0,1]) &\to L^{p_1}([0,1]) \\ Tf &= f \end{aligned}$$

has $\|T\|_{op} = 1$.

This means that the inclusions are in the reverse direction compared to the inclusions among ℓ^p spaces. One easy case is to see that $L^\infty([0,1]) \subseteq L^1([0,1])$ because

$$\int_0^1 |f(x)|\, dx$$

will clearly be finite if the integrand is bounded.

The general case $L^{p_2}([0,1]) \subseteq L^{p_1}([0,1])$ (and the fact that the inclusion operator has norm at most 1) follows from Hölders inequality by taking one of the functions to be the constant 1 and a suitable value of p. At least this works if $p_2 < \infty$.

$$\begin{aligned} \|f\|_{p_1}^{p_1} &= \int_0^1 |f(x)|^{p_1}\, dx \\ &= \int_0^1 |f(x)|^{p_1} 1\, dx \\ &\leq \left(\int_0^1 (|f(x)|^{p_1})^p\, dx \right)^{1/p} \left(\int_0^1 1^q\, dx \right)^{1/q} \\ &\qquad \text{(with } \frac{1}{p} + \frac{1}{q} = 1\text{)} \\ &= \left(\int_0^1 |f(x)|^{p_1 p}\, dx \right)^{1/p} = \|f\|_{p_1 p}^{p_1} \end{aligned}$$

To make $p_1 p = p_2$ we take $p = p_2/p_1$, which is allowed as $p_2 > p_1$. We get

$$\|f\|_{p_1}^{p_1} \leq \|f\|_{p_2}^{p_1}$$

and so we have $\|f\|_{p_1} \leq \|f\|_{p_2}$.

If $f \in L^{p_2}([0,1])$, which means that $\|f\|_{p_2} < \infty$, we see that $f \in L^{p_1}([0,1])$. So we have inclusion as claimed, but also the inequality $\|f\|_{p_1} \leq \|f\|_{p_2}$ tells us that $\|T\|_{op} \leq 1$. Taking f to be the constant function 1, we see that $\|f\|_{p_1} = 1 = \|f\|_{p_2}$ and so $\|T\|_{op}$ cannot be smaller than 1.

When $p_2 = \infty$ there is a simpler argument based on

$$\int_0^1 |f(x)|^{p_1}\, dx \leq \int_0^1 \|f\|_\infty^{p_1}\, dx = \|f\|_\infty^{p_1}$$

to show $L^\infty([0,1]) \subseteq L^{p_1}([0,1])$ and the inclusion has norm at most 1. Again the constant function 1 shows that $\|T\|_{op} = 1$ in this case.

(iii) If (X, Σ, μ) is a finite measure space (that is if $\mu(X) < \infty$) then we have a somewhat similar result to what we have for $L^p([0,1])$. The inclusions go the same way, but the inclusion operators can gave norm different from 1. If $1 \leq p_1 < p_2 \leq \infty$, then $L^{p_2}(X, \Sigma, \mu) \subseteq L^{p_1}(X, \Sigma, \mu)$ and the inclusion operator

$$\begin{aligned} T\colon L^{p_2}(X, \Sigma, \mu) &\to L^{p_1}(X, \Sigma, \mu) \\ Tf &= f \end{aligned}$$

has

$$\|T\|_{op} = \mu(X)^{(1/p_1)-(1/p_2)}.$$

The proof is quite similar to the previous case, but the difference comes from the fact that $\int_X 1\, d\mu(x) = \mu(X)$ and this is not necessarily 1. (When $\mu(X) = 1$ we are in a probability space.) So when we use Hölders inequality, a constant will come out and the resulting estimate for $\|T\|_{op}$ is the value above. Again if you look at the constant function 1, you see that $\|T\|_{op} \geq \mu(X)^{(1/p_1)-(1/p_2)}$.

(iv) When we look at the different $L^p(\mathbb{R})$ spaces, we find that the arguments above don't work. The argument with Hölders inequality breaks down because the constant function 1 has integral ∞ and the argument that worked for ℓ^p does not go anywhere either.

Looking at the extreme cases of $p = 1$ and $p = \infty$, there is no reason to conclude that $|f(x)|$ bounded implies $\int_{-\infty}^{\infty} |f(x)|\,dx$ should be finite. And on the other hand functions where the integral is finite can be unbounded.

This actually turns out to be the case. We can find examples of function in $L^{p_1}(\mathbb{R})$ but not in $L^{p_2}(\mathbb{R})$ for any values of $1 \leq p_1, p_2 \leq \infty$ where $p_1 \neq p_2$.

To see this we consider two examples of function $f_\alpha, g_\alpha \colon \mathbb{R} \to \mathbb{K}$, given as

$$f_\alpha(x) = \frac{e^{-x^2}}{|x|^\alpha}, \quad g_\alpha(x) = \frac{1}{1 + |x|^\alpha}$$

(where $\alpha > 0$).

When checking to see if $f_\alpha \in L^p(\mathbb{R})$ (for $p < \infty$) or not we end up checking if

$$\int_0^1 \frac{1}{x^{p\alpha}}\,dx < \infty.$$

The reason is that $\int_{-\infty}^{\infty} |f_\alpha(x)|^p\,dx = 2\int_0^{\infty} |f_\alpha(x)|^p\,dx$ (even function) and the e^{-x^2} guarantees $\int_1^{\infty} |f_\alpha(x)|^p\,dx < \infty$. In the range $0 < x < 1$ the exponential term is neither big nor small and does not affect convergence of the integral. The condition comes down to $p\alpha < 1$ or $\alpha < 1/p$. The case $p = \infty$ (and $1/p = 0$) fits into this because (when $\alpha > 0$) $f_\alpha(x) \to \infty$ as $x \to 0$ and so $f_\alpha \notin L^\infty(\mathbb{R})$.

For $g_\alpha \in L^p(\mathbb{R})$ to hold we end up with the condition

$$\int_1^{\infty} \frac{1}{x^{p\alpha}}\,dx < \infty$$

(as $g_\alpha(x)$ is within a constant factor of $1/x^{p\alpha}$ when $x \geq 1$) which is true if $p\alpha > 1$ or $\alpha > 1/p$.

Thus if $1 \leq p_1 < p_2 \leq \infty$ and we choose α in the range

$$\frac{1}{p_2} < \alpha < \frac{1}{p_1}$$

we find

$$f_\alpha \in L^{p_1}(\mathbb{R}), f_\alpha \notin L^{p_2}(\mathbb{R}), g_\alpha \in L^{p_2}(\mathbb{R}), g_\alpha \notin L^{p_1}(\mathbb{R}),$$

which shows that neither $L^{p_1}(\mathbb{R}) \subseteq L^{p_2}(\mathbb{R})$ nor $L^{p_2}(\mathbb{R}) \subseteq L^{p_1}(\mathbb{R})$ is valid.

(v) You might wonder why in the case of (X, Σ, μ) being $X = \mathbb{N}$ with counting measure, and so $\mu(X) = \infty$, we do get $\ell^{p_1} \subseteq \ell^{p_2}$ but that this does not work with $X = \mathbb{R}$ (where also $\mu(X) = \infty$).

The difference can be explained by the fact that $\mathbb{N}$ is what is called as an atomic measure space. The singleton sets are of strictly positive measure and cannot be subdivided. On the other hand $\mathbb{R}$ is what is called 'purely nonatomic'. There are no 'atoms' (sets of positive measure which cannot be written as the union of two disjoint parts each of positive measure).

1.8.7 Definition. Two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are *isomorphic* if there exists a vector space isomorphism $T\colon E \to F$ which is also a homeomorphism.

We say that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are called *isometrically isomorphic* if there exists a vector space isomorphism $T\colon E \to F$ which is also isometric ($\|Tx\|_F = \|x\|_E$ for all $x \in E$).

1.8.8 Remark. From Theorem 1.8.1, we can see that both T and T^{-1} are bounded if $T\colon E \to F$ is both a vector space isomorphism and a homeomorphism. Thus there exist constants $M_1 = \|T\|_{op}$ and $M_2 = \|T^{-1}\|_{op}$ so that

$$\frac{1}{M_2}\|x\|_E \le \|Tx\|_F \le M_1\|x\|_E \qquad (\forall x \in E)$$

This means that an isomorphism almost preserves distances (preserves them within fixed ratios)

$$\frac{1}{M_2}\|x - y\|_E \le \|Tx - Ty\|_F \le M_1\|x - y\|_E$$

as well as being a homeomorphism (which means more or less preserving the topology).

Another way to think about it is that if we transfer the norm from F to E via the map T to get a new norm on E given by

$$|||x||| = \|Tx\|_F$$

then we have

$$\frac{1}{M_2}\|x\|_E \le |||x||| \le M_1\|x\|_E.$$

1.8.9 Theorem. *If $(E, \|\cdot\|_E)$ is a finite dimensional normed space of dimension n, then E is isomorphic to $\mathbb{K}^n$ (with the standard Euclidean norm).*

Proof. Let $v_1, v_2, \ldots, v_n$ be a vector space basis for E and define $T\colon \mathbb{K}^n \to E$ by

$$T(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i v_i.$$

From standard linear algebra we know that T is a vector space isomorphism, and what we have to show is that T is also a homeomorphism.

To show first that T is continuous (or bounded) consider (for fixed $1 \leq j \leq n$) the map $T_j\colon \mathbb{K}^n \to E$ given by

$$T_j(x_1, x_2, \ldots, x_n) = x_j v_j.$$

We have $\|T_j(x)\|_E = |x_j|\|v_j\|_E \leq \|v_j\|_E \|x\|_2$ since $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \geq |x_j|$. From the triangle inequality we can deduce

$$\|Tx\|_E = \left\| \sum_{j=1}^{n} T_j(x) \right\|_E \leq \sum_{j=1}^{n} \|T_j(x)\|_E \leq \left(\sum_{j=1}^{n} \|v_j\|_E \right) \|x\|_E$$

establishing that T is bounded (with operator norm at most $\sum_{j=1}^{n} \|v_j\|_E$).

Now consider the unit sphere

$$S = \{x \in \mathbb{K}^n : \|x\|_2 = 1\},$$

which we know to be a compact subset of $\mathbb{K}^n$ (as it is closed and bounded). Since T is continuous, $T(S)$ is a compact subset of E. Thus $T(S)$ is closed in E (because E is Hausdorff). Since T is bijective (and $0 \notin S$) we have $0 = T(0) \notin T(S)$. Thus there exists $r > 0$ so that

$$\{y \in E : \|y - 0\|_E = \|y\|_E < r\} \subset E \setminus S.$$

Another way to express this is

$$x \in \mathbb{K}^n, \|x\|_2 = 1 \Rightarrow \|Tx\|_E \geq r.$$

Scaling arbitrary $x \in \mathbb{K}^n \setminus \{0\}$ to get a unit vector $x/\|x\|_2$ we find that

$$\left\| T\left(\frac{1}{\|x\|_2} x \right) \right\|_E = \left\| \frac{1}{\|x\|_2} Tx \right\|_E = \frac{1}{\|x\|_2} \|Tx\|_E \geq r$$

and so $\|Tx\|_E \geq r\|x\|_2$. This holds for $x = 0$ also. So for $y \in E$, we can take $x = T^{-1}y$ to get $\|y\|_E \geq r\|T^{-1}y\|_2$, or

$$\|T^{-1}y\|_2 \leq \frac{1}{r}\|y\|_E \quad (\forall y \in E).$$

Thus T^{-1} is bounded. $\square$

1.8.10 Corollary. *If $(E, \|\cdot\|_E)$ is a finite dimensional normed space and $T\colon E \to F$ is any linear transformation with values in any normed space $(F, \|\cdot\|_F)$, then T is continuous.*

Proof. Chose an isomorphism $S\colon \mathbb{K}^n \to E$ (which exists by Theorem 1.8.9). Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{K}^n$ (so that $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, o)$, etc.) and $w_i = (T \circ S)(e_i)$. Then

$$(T \circ S)(x_1, x_2, \ldots, x_n) = (T \circ S)\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i w_i$$

and as in the proof of Theorem 1.8.9 we can show that $T \circ S$ is bounded (in fact $\|T \circ S\|_{op} \leq \sum_{i=1}^n \|w_i\|_F$). So $T = (T \circ S) \circ S^{-1}$ is a composition of two continuous linear operators, and is therefore continuous. □

1.8.11 Example. The Corollary means that for finite dimensional normed spaces, continuity or boundedness of linear transformations is automatic. For that reason we concentrate most on infinite dimensional situations and we always restrict our (main) attention to bounded linear operators.

Although in finite dimensions boundedness is not in question, that only means that there exists some finite bound. There are still questions (which have been considered in much detail by now in research) about what the best bounds are. These questions can be very difficult in finite dimensions. Part of the motivation for this study is that a good understanding of the constants that arise in finite dimensions, and how they depend on increasing dimension, can reveal insights about infinite dimensional situations.

Consider the finite dimensional version of ℓ^p, that is $\mathbb{K}^n$ with the norm $\|\cdot\|_p$ given by

$$\|(x_1, x_2, \ldots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

for $1 \leq p < \infty$ and $\|(x_1, x_2, \ldots, x_n)\|_\infty = \max_{1 \leq i \leq n} |x_i|$. We denote this space by ℓ^p_n, with the n for dimension.

Now, all the norms $\|\cdot\|_p$ are equivalent on $\mathbb{K}^n$, that is given p_1 and p_2 there are constants $m, M > 0$ so that

$$m\|x\|_{p_1} \leq \|x\|_{p_2} \leq M\|x\|_{p_1}$$

holds for all $x \in \mathbb{K}^n$. We can identify the constants more precisely. Say $1 \leq p_1 < p_2 \leq \infty$. Then we know from the infinite dimensional inequality for ℓ^p spaces (applied to the finitely nonzero sequence $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$) that $\|x\|_{p_2} \leq \|x\|_{p_1}$.

We could also think of ℓ^p_n as a L^p space where we take $X = \{1, 2, \ldots, n\}$ with counting measure μ. From Examples 1.8.6 (iii) we know also $\|x\|_{p_1} \leq n^{(1/p_1)-(1/p_2)}\|x\|_{p_2}$. In summary

$$\|x\|_{p_2} \leq \|x\|_{p_1} \leq n^{(1/p_1)-(1/p_2)}\|x\|_{p_2}$$

One can visualise this geometrically (say in $\mathbb{R}^2$ to make things easy) in terms of the shapes of the open unit balls

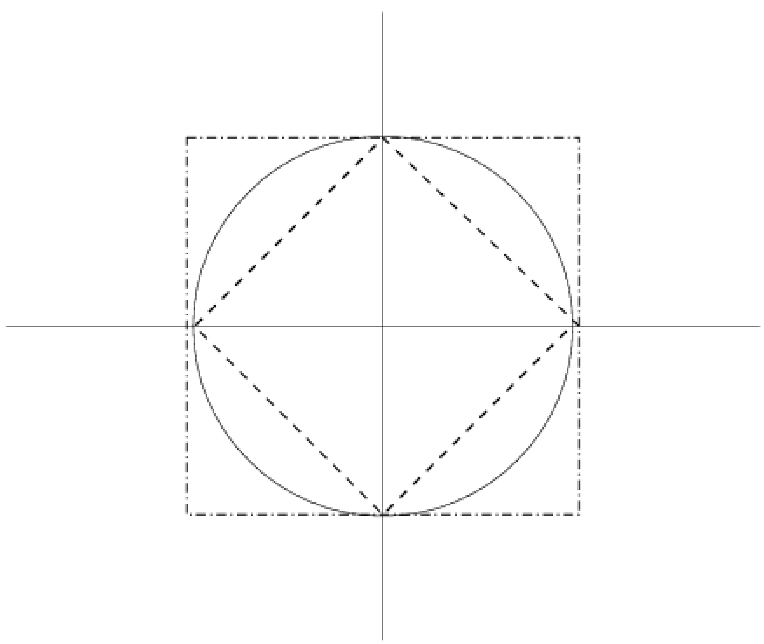
$$B_{\ell^p_n} = \{x \in \ell^p_n : \|x\|_p < 1\}$$

The inequality above means

$$B_{\ell^{p_1}_n} \subseteq B_{\ell^{p_2}_n} \subseteq n^{(1/p_1)-(1/p_2)} B_{\ell^{p_1}_n}.$$

For $p = 2$ we get a familiar round ball or circular disc, but for other p the shapes are different. For $p = 1$ we get a diamond (or square) with corners $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. This is because the unit ball is specified by $|x_1| + |x_2| < 1$. So in the positive quadrant this comes to $x_1 + x_2 < 1$, or (x_1, x_2) below the line joining $(1, 0)$ and $(0, 1)$.

The ball of (real) ℓ^∞_2 on the other hand is given by $\max(|x_1|, |x_2|) < 1$ or $-1 \leq x_i \leq 1$. So its unit ball is a square with the 4 corners $(\pm 1, \pm 1)$. Here is an attempt at drawing the unit balls for ℓ^2_2, ℓ^1_2 and ℓ^∞_2 on the one picture.



For example the fact that

$$B_{\ell_n^1} \subseteq B_{\ell_n^2} \subseteq n^{(1/1)-(1/2)} B_{\ell_n^1} = \sqrt{n} B_{\ell_n^1}$$

means for $n = 2$ that the diamond is contained in the circle and the circle is contained in the diamond expanded by a factor $\sqrt{2}$.

1.8.12 Corollary. *Finite dimensional normed spaces are complete.*

Proof. If E is an n-dimensional normed space, then there is an isomorphism $T \colon \mathbb{K}^n \to E$. As $\mathbb{K}^n$ is complete and T^{-1} is uniformly continuous, it follows from Proposition 1.4.16 that E is complete. □

1.9 Open mapping, closed graph and uniform boundedness theorems

1.9.1 Theorem (Open Mapping Theorem)**.** *Let* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *be Banach spaces and* $T \colon E \to F$ *a surjective bounded linear operator. Then there exists* $\delta > 0$ *so that*

$$T(B_E) \supseteq \delta B_F$$

where $B_E = \{x \in E : \|x\|_E < 1\}$ *and* $B_F = \{y \in F : \|y\|_F < 1\}$ *are the open unit balls of* E *and* F*.*

Moreover, if $U \subseteq E$ *is open, then* $T(U)$ *is open (in* F*).*

Proof. First we show that the second statement (the 'open mapping' part) follows from the first.

It is handy to note that all (open) balls $B(x_0, r) = \{x \in E : \|x - x_0\|_E < r\}$ in a normed space (or Banach space) E can be related to the unit ball B_E as follows

$$B(x_0, r) = x_0 + rB_E$$

(where $rB_E = \{rx : x \in B_E\}$ is a stretching of B_E be the factor r and $x_0 + rB_E = \{x_0 + rx : x \in B_E\}$ is the translate of rB_E by x_0).

So if $U \subseteq E$ is open and $y_0 \in T(U)$ then we know $y_0 = T(x_0)$ for some $x_0 \in U$. Since U is open, there is $r > 0$ with $x_0 + rB_E = B(x_0, r) \subseteq U$. It follows from $T(B_E) \supseteq \delta B_F$ and linearity of T that

$$T(U) \supseteq T(x_0 + rB_E) = T(x_0) + rT(B_E) \supseteq y_0 + r\delta B_F = B(y_0, r\delta).$$

Thus y_0 is an interior point of $T(U)$. As this is true of all $y_0 \in T(U)$, we have shown that $T(U)$ is open.

Now to prove the first assertion, let $V_r = T(rB_E) = rT(B_E)$ for $r > 0$. Because T is surjective and $E = \bigcup_{n=1}^\infty nB_E$, we have

$$F = \bigcup_{n=1}^\infty T(nB_E) = \bigcup_{n=1}^\infty V_n.$$

By the Baire category theorem (and this is where we use the assumption that F is complete) we cannot have V_n nowhere dense for all n. (See §1.6 for an explanation of what the Baire category theorem says, and its proof.) That means there is n so that the closure $\bar{V}_n$ has nonempty interior, or in other words there is $y_0 \in \bar{V}_n$ and $r > 0$ with $B(y_0, r) = y_0 + rB_F \subseteq \bar{V}_n$.

By changing y_0 and reducing r we can assume $y_0 \in V_n$ (and not just y_0 in the closure). The idea is that there must be $y_1 \in V_n \cap B(y_0, r/2)$ and then $B(y_1, r/2) \subseteq B(y_0, r) \subseteq \bar{V}_n$. Write $y_1 = T(x_0)$ for $x_0 \in nB_E$. Then we claim that $\bar{V}_{2n} \supseteq (r/2)B_F$. The argument is that $x_0 \in nB_E$ implies

$$2nB_E \supseteq nB_E - x_0$$

and so

$$V_{2n} \supseteq T(nB_E - x_0) = T(nB_E) - T(x_0) = V_n - y_1$$

Taking closures, we get

$$\bar{V}_{2n} \supseteq \bar{V}_n - y_1 \supseteq (y_1 + (r/2)B_F) - y_1 = (r/2)B_F.$$

The fact that the closure of $V_n - y_1$ is the same as $\bar{V}_n - y_1$ follows from the fact that the translation map $y \mapsto y - y_1 \colon F \to F$ is distance preserving (and so a homeomorphism) of F onto itself (with inverse the translation $y \mapsto y + y_1$). Homeomorphisms preserve all topological things, and map closures to closures.

For $k \in \mathbb{N}$ we have $\bar{V}_{kn} = \overline{(kV_n)} = k\bar{V}_n$ (one can check that because $y \mapsto ky$ is a homeomorphism with inverse $y \mapsto (1/k)y$). Thus, if k big enough so that $kr/2 > 1$ we have

$$\bar{V}_{kn} = k\bar{V}_n \supseteq k\frac{r}{2}B_F \supseteq B_F$$

Let $N = kn$. Thus we have $\bar{V}_N \supseteq B_F$. What we *claim* is that $V_{3N} \supseteq B_F$ (no closure now on the V_{3N}).

From $\bar{V}_N \supseteq B_F$ we deduce, for $j \in \mathbb{N}$,

$$\bar{V}_{N/j} = \overline{\frac{1}{j}V_N} = \frac{1}{j}\bar{V}_N \supseteq \frac{1}{j}B_F$$

Starting with any $y \in B_F$ we must have $x_1 \in NB_E$ (so $T(x_1) \in V_N$) with

$$\|y - T(x_1)\|_F < \frac{1}{2}$$

or

$$y - T(x_1) \in \frac{1}{2}B_F \subseteq \bar{V}_{N/2}.$$

We can then find $x_2 \in (N/2)B_E$ (so $T(x_2) \in V_{N/2}$) with

$$\|(y - T(x_1)) - T(x_2)\|_F < \frac{1}{2^2}$$

or

$$y - T(x_1) - T(x_2) = y - (T(x_1) + T(x_2)) \in \frac{1}{2^2}B_F \subseteq \bar{V}_{N/2^2}.$$

By induction, we can find $x_1, x_2, \dots$ with $x_j \in (N/2^{j-1})B_E$ and

$$y - \sum_{i=1}^{j} T(x_i) \in \frac{1}{2^j}B_F \subseteq \bar{V}_{N/2^j}.$$

With this construction we have an absolutely convergent series $\sum_{j=1}^{\infty} x_j$ in E, because

$$\sum_{j=1}^{\infty} \|x_j\|_E \leq \sum_{j=1}^{\infty} \frac{N}{2^{j-1}} = 2N < 3N.$$

Now using completeness of E (remember we used completeness of F earlier) we know from Proposition 1.7.8 that $\sum_{j=1}^{\infty} x_j$ converges in E. That says there is $x \in E$ with $x = \lim_{j\to\infty} \sum_{i=1}^{j} x_i$ and in fact we know

$$\|x\|_E \leq \lim_{j\to\infty} \left\| \sum_{i=1}^{j} x_i \right\|_E \leq \lim_{j\to\infty} \sum_{i=1}^{j} \|x_i\|_E \leq 2N < 3N$$

because the closed ball $\bar{B}(0, 2N)$ is closed in E and all the partial sums $\sum_{i=1}^{j} x_i$ are inside $\bar{B}(0, 2N)$.

As T is linear and continuous, we also have

$$Tx = \lim_{j\to\infty} T\left(\sum_{i=1}^{j} x_i\right) = \lim_{j\to\infty} \sum_{i=1}^{j} T(x_i) = y$$

because

$$\left\| y - \sum_{i=1}^{j} T(x_i) \right\|_F \leq \frac{1}{2^j} \to 0$$

as $j \to \infty$.

So we have $y = T(x) \in V_{3N} = T(3NB_E)$. Since this is so for all $y \in B_F$, we have $B_F \subseteq V_{3N}$ as claimed. It follows that

$$T(B_E) \subseteq \frac{1}{3N} B_F,$$

which is the result with $\delta = 1/(3N) > 0$. □

1.9.2 Corollary. *If E, F are Banach spaces and $T\colon E \to F$ is a bounded linear operator that is also bijective, then T is an isomorphism.*

Proof. By the Open Mapping theorem T is automatically an open map, that is $U \subseteq E$ open implies $T(U) \subseteq F$ open. But, since T is a bijection the forward image $T(U)$ is the same as the inverse image $(T^{-1})^{-1}(U)$ of U under the inverse map T^{-1}.

Thus the open mapping condition says that T^{-1} is continuous. □

1.9.3 Theorem (Closed graph theorem)**.** *If E, F are Banach spaces and $T\colon E \to F$ is a linear transformation, then T is bounded if and only if its 'graph'*

$$\{(x, y) \in E \times F : y = Tx\}$$

is a closed subset of $E \times F$ in the product topology.

Proof. It is quite easy to check that the graph must be closed if T is bounded (continuous). The product topology is a metric space topology as E and F are metric spaces, and in fact arises from a norm on $E \oplus F$. (See §A.2 and §A.3.) For definiteness we take the norm

$$\|(x, y)\|_1 = \|x\| + \|y\|$$

on $E \oplus F$, or we could write $E \oplus_1 F$. So to show the graph is closed, suppose we have a sequence (x_n, y_n) in the graph that converges in $E \times F$ to some limit (x, y). We claim that (x, y) must be in the graph, that is that $y = Tx$.

What we know is that $y_n = Tx_n$ for each n (since (x_n, y_n) is in the graph). As $(x_n, y_n) \to (x, y)$ in the product topology, it follows that $x_n \to x$ and $y_n \to y$ (as $n \to \infty$). (This is rather easy as $\|x_n - x\| \leq \|(x_n, y_n) - (x, y)\|_1 = \|(x_n - x, y_n - y)\|_n \to 0$ as $n \to \infty$.) But T is continuous and so $T(x_n) \to T(x)$. Since $T(x_n) = y_n$, that means $y_n \to T(x)$. As $y_n \to y$ also (because $\|y_n - y\| \leq \|(x_n, y_n) - (x, y)\|_1$) and in metric spaces sequences can have at most one limit, we have $T(x) = y$ and (x, y) is in the graph — as we wanted to show.

Now assume that the graph is closed. Our proof that T must be continuous relies on the open mapping theorem. We introduce a new norm on E by

$$|||x||| = \|x\|_E + \|Tx\|_F.$$

It is not at all difficult to check that this is indeed a norm:

$$\begin{aligned}
|||x_1 + x_2||| &= \|x_1 + x_2\|_E + \|T(x_1 + x_2)\|_F \\
&= \|x_1 + x_2\|_E + \|T(x_1) + T(x_2)\|_F \\
&\leq \|x_1\|_E + \|x_2\|_E + \|T(x_1)\|_F + \|T(x_2)\|_F \\
&= |||x_1||| + |||x_2||| \\
|||\lambda x||| &= \|\lambda x\|_E + \|T(\lambda x)\|_F \\
&= \|\lambda x\|_E + \|\lambda Tx\|_F \\
&= |\lambda|\|x\|_E + |\lambda|\|Tx\|_F \\
&= |\lambda| |||x||| \\
|||x||| = 0 &\Rightarrow \|x\|_E = 0 \\
&\Rightarrow x = 0
\end{aligned}$$

We consider the identity mapping id: $(E, |||\cdot|||) \to (E, \|\cdot\|_E)$ which is clearly linear, bijective and bounded since

$$\|\text{id}(x)\|_E = \|x\|_E \leq |||x|||.$$

In order to apply the open mapping theorem we need to know that $(E, |||\cdot|||)$ is a Banach space (that is complete) and in doing that the hypothesis that the graph is closed will be used. Once we have verified completeness, the open mapping theorem guarantees that the inverse map $\text{id}^{-1}\colon (E, \|\cdot\|_E) \to (E, |||\cdot|||)$ is continuous. Therefore its is bounded, or there exists $M \geq 0$ so that

$$|||x||| \leq M\|x\|_E$$

holds for all $x \in E$. That implies $\|Tx\|_F \leq \|\!|x|\!\| \leq M\|x\|_E$ and so T bounded as we need.

It remains to verify the completeness of $(E, \|\!|\cdot|\!\|)$. Take a Cauchy sequence $(x_n)_{n=1}^\infty$ in E with respect to the metric arising from $\|\!|\cdot|\!\|$. Then for $\varepsilon > 0$ given and n, m large enough

$$\|\!|x_n - x_m|\!\| < \varepsilon.$$

But that implies both $\|x_n - x_m\|_E < \varepsilon$ and $\|T(x_n) - T(x_m)\|_F < \varepsilon$. This means $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $(E, \|\cdot\|_E)$ and $(T(x_n))_{n=1}^\infty$ is a Cauchy sequence in $(F, \|\cdot\|_F)$. As both E and F are complete there exists $x = \lim_{n\to\infty} x_n \in E$ and $y = \lim_{n\to\infty} T(x_n) \in F$. Using 3.39 (ii) we see that

$$(x_n, T(x_n)) \to (x, y)$$

in the product topology of $E \times F$. But each $(x_n, T(x_n))$ is in the graph of T and we are assuming that the graph is closed. So the limit (x, y) is also in the graph, or $y = Tx$. It follows that

$$\lim_{n\to\infty} \|\!|x_n - x|\!\| = \lim_{n\to\infty} \|x_n - x\|_E + \|T(x_n) - Tx\|_F = \lim_{n\to\infty} \|x_n - x\|_E + \|y_n - y\|_F = 0$$

and that means $x_n \to x$ in $(E, \|\!|\cdot|\!\|)$. WE have shown that each Cauchy sequence in $(E, \|\!|\cdot|\!\|)$ converges. So $(E, \|\!|\cdot|\!\|)$ is complete.

As noted above, the result follows then from the open mapping theorem. □

1.9.4 Theorem (Uniform boundedness principle)**.** *Let E be a Banach space and F a normed space. Let $T_i \colon E \to F$ be bounded linear operators for $i \in I =$ some index set. Assume*

$$\sup_{i\in I} \|T_i(x)\| < \infty \text{ for each } x \in E$$

(which can be stated as the family of operators being pointwise uniformly bounded).

Then

$$\sup_{i\in I} \|T_i\| < \infty$$

(uniform boundedness in norm).

Proof. The proof relies on the Baire category theorem in E.

Let

$$\begin{aligned} W_n &= \{x \in E : \sup_{i\in I} \|T_i x\| \leq n\} \\ &= \bigcap_{i\in I} \{x \in E : \|T_i x\| \leq n\}. \end{aligned}$$

Then W_n is closed as it is the intersection of closed sets. ($\{x \in E : \|T_i x\| \leq n\}$ is the inverse image under T_i of the closed ball $\{y \in F : \|y\| \leq n\}$, and T_i is continuous. So the inverse image is closed.)

The hypothesis implies that each $x \in E$ belongs in W_n for some n, or that $\bigcup_{n=1}^{\infty} W_n = E$. By the Baire category theorem, there is some n where W_n is not nowhere dense. Fix this n. As W_n is closed, W_n not nowhere dense means that W_n has nonempty interior. So there is x_0 and $r > 0$ so that

$$B(x_0, r) = x_0 + rB_E \subseteq W_n.$$

Thus

$$\|T_i(x_0 + rx)\| = \|T_i(x_0) + rT_i(x)\| \leq n$$

holds for all $x \in B_E$. It follows that

$$r\|T_i(x)\| - \|T_i(x_0)\| \leq n \quad (\forall x \in B_E)$$

and so

$$\|T_i(x)\| \leq \frac{1}{r}\|T_i(x_0)\| + \frac{n}{r} \leq \frac{2n}{r} \quad (\forall x \in B_E)$$

since $x_0 \in W_n \Rightarrow \|T_i(x_0)\| \leq n$ for all $i \in I$. Take $\sup_{x \in B_E}$ to get

$$\|T_i\| \leq \frac{2n}{r}$$

and as this is true for each $i \in I$, we have $\sup_{i \in I} \|T_i\| \leq 2n/r < \infty$ as required. □

1.9.5 Remark. The open mapping, closed graph and uniform boundedness principle are all considered to be fundamental theorems in functional analysis. We will postpone giving examples of their use until we discuss Fourier series later.

There is at least one more theorem of central importance, the Hahn Banach theorem, and again we leave that for a while.

A Appendix

A.1 Uniform convergence

We recall here some facts about unform convergence of sequences of functions (with values in a metric space).

A.1.1 Definition. Let X be a topological space and (Y, d) a metric space. Let $f_n\colon X \to Y$ be functions ($n = 1, 2, 3, \ldots$) and $f_0\colon X \to Y$ another function.

Then we say that the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on X to f_0 if the following is true

for each $\varepsilon > 0$ there exists $N > 0$ so that

$$n \geq N, x \in X \Rightarrow d(f_n(x), f_0(x)) < \varepsilon.$$

A.1.2 Proposition. *('uniform limits of continuous functions are continuous') Let X be a topological space and (Y, d) a metric space. If $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions $f_n\colon X \to Y$ that converges uniformly on X to $f_0\colon X \to Y$, then f_0 is continuous.*

Proof. Fix $x_0 \in X$ and our aim will be to show that f is continuous at x_0. So take a neighbourhood N^Y of $y_0 = f(x_0) \in Y$, and our aim is to show $f^{-1}(N^Y)$ is a neighbourhood of $x_0 \in X$.

There is $\varepsilon > 0$ so that $B(y_0, \varepsilon) \subseteq N^Y$. By uniform convergence we can find n_0 so that

$$n \geq n_0, x \in X \Rightarrow d(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

Fix $n = n_0$. By continuity of f_n at x_0,

$$N^X = f_n^{-1}\left(B(f_n(x_0), \frac{\varepsilon}{3}\right) = \left\{x \in X : d(f_n(x), f_n(x_0)) < \frac{\varepsilon}{3}\right\}$$

is a neighbourhood of $x_0 \in X$. For $x \in N^X$ we have

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

This implies $f(x) \in B(f(x_0), \varepsilon) \subseteq N^Y \forall x \in N^X$ or that $N^X \subseteq f^{-1}(N^Y)$. So $f^{-1}(N^Y)$ is a neighbourhood of $x_0 \in X$.

This shows f continuous at $x_0 \in X$. As x_0 is arbitrary, f is continuous. □

A.2 Products of two metric spaces

Given two metric space (X, d_X) and (Y, d_Y), we can make the cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$ into a metric space in a number of ways. For example we can define a metric

d_∞ on $X \times Y$ by the rule

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

It is not very difficult to check that his does define a metric. For example the triangle inequality

$$d_\infty((x_1, y_1), (x_3, y_3)) \leq d_\infty((x_1, y_1), (x_2, y_2)) + d_\infty((x_2, y_2), (x_3, y_3)),$$

we start by

$$\begin{aligned} d_\infty((x_1, y_1), (x_3, y_3)) &= \max(d_X(x_1, x_3), d_Y(y_1, y_3)) \\ &\leq \max(d_X(x_1, x_2) + d_X(x_2, x_3), d_Y(y_1, y_2) + d_X(y_2, y_3)) \end{aligned}$$

(since $d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3)$ and $d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_X(y_2, y_3)$ by the triangle inequalities for d_X and D_y). Then use the triangle inequality for the norm $\|\cdot\|_\infty$ on $\mathbb{R}^2$.

However, the most familiar example of a product is perhaps $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and the familiar Euclidean metric on $\mathbb{R}^2$ is not the one we get by using d_∞ with $d_{\mathbb{R}}(x_1, x_2) = d_X(x_1, x_2) = |x_1 - x_2|$, $d_Y = d_X$. Motivated by that example, we might prefer to think of a different metric d_2 on the product of two metric spaces (X, d_X) and (Y, d_Y) where we take

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2} = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|_2.$$

The proof that d_2 is a metric is not that different from the proof that d_∞ is a metric on $X \times Y$.

More generally we might take p in the range $1 \leq p < \infty$ and define d_p on $X \times Y$ by

$$d_p((x_1, y_1), (x_2, y_2)) = ((d_X(x_1, x_2))^p + (d_Y(y_1, y_2))^p)^{1/p} = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|_p.$$

Again it is not that hard to show that d_p is a metric.

We have now an embarrassment of definitions in that we have infinitely many choices $(X \times Y, d_p)$ with $1 \leq p \leq \infty$. However, all these metrics are somewhat comparable because if $1 \leq p_1 \leq p_2 \leq \infty$, then

$$\|a\|_{p_2} \leq \|a\|_{p_1} \leq 2\|a\|_{p_2} \quad (a \in \mathbb{R}^2)$$

(see Example 1.8.11 with $n = 2$, where a more precise constant is given). It follows that

$$d_{p_2}((x_1, y_1), (x_2, y_2)) \leq d_{p_1}((x_1, y_1), (x_2, y_2)) \leq 2d_{p_2}((x_1, y_1), (x_2, y_2))$$

and as a consequence the open sets in $(X \times Y, d_p)$ are the same for every p. So the continuous functions with domains $X \times Y$ and values in some topological space Z are the same no matter what matric we use. The same is true of functions $f \colon Z \to X \times Y$ with values in $X \times Y$. Moreover the sequences $((x_n, y_n))_{n=1}^\infty$ that converge in $(X \times Y, d_p)$ do not depend on the value of p we use.

It is possible to describe the topology on $X \times Y$ arising from any one of these metrics d_p in a way that uses only topology (open sets). We will not do that at this stage. The topology on $X \times Y$ arising from any one of the d_p is called the *product topology*.

A.3 Direct sum of two normed spaces

If we start with normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, then we can make $E \times F$ into a vector space by defining vector space operations as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad (\text{for } (x_1, y_1), (x_2, y_2) \in E \times F)$$

$$\lambda(x, y) = (\lambda x, \lambda y) \quad (\text{for } \lambda \in \mathbb{K}, (x, y) \in E \times F).$$

One can check in a straightforward way that this make $E \times F$ into a vector space, and the usual notation for this is $E \oplus F$ (called the *direct sum* of E and F).

Inside $E \oplus F$ there is a subspace

$$E \oplus \{0\} = \{(x, 0) : x \in E\}$$

that behaves just like E (is isomorphic as a vector space to E) and another subspace $\{0\} \oplus F = \{(0, y) : y \in F\}$ that is a copy of F. Every $(x, y) \in E \oplus F$ can be expressed in a unique way as

$$(x, y) = (x, 0) + (0, y)$$

as a sum of an element of $E \oplus \{0\}$ and an element of $\{0\} \oplus F$.

So far this is about vector spaces and so belongs to linear algebra. To make $E \oplus F$ a normed space, we have at least the same sort of choices for norms as we discussed for metrics in a product metric space. For $1 \leq p \leq \infty$ we can define a norm on $E \oplus F$ by

$$\|(x, y)\|_p = \|(\|x\|_E, \|y\|_F)\|_p = (\|x\|_E^p + \|y\|_F^p)^{1/p}$$

(in terms of the p-norm of $\mathbb{R}^2$). It is not hard to check that this is a norm and that the distance arising from this norm is the same as the metric d_p arising from the metrics d_E and d_F given by the norms on E and F.

It is common to write $E \oplus_p F$ for $E \oplus F$ with the norm $\|\cdot\|_p$.