UNIVERSITY OF DUBLIN

sample

TRINITY COLLEGE

Faculty of Engineering, Mathematics
and Science

school of mathematics

JS & SS Mathematics **Trinity Term 2009**
SAMPLE PAPER

Course 321 — Functional Analysis

Dr. R. M. Timoney

Credit will be given for the best 6 questions answered.

The actual exam has 8 questions total (and NOT 9 as here).

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (a) In a metric space (X, d), explain what is meant by the *interior* and *closure* of a subset $S \subset X$.

   (b) Give an example of an open ball $B(x, r)$ (or radius r, centred at $x \in X$) in a metric space (X, d) such that the closure of $B(x, r)$ is different from the closed ball $\bar{B}(x, r)$ with the same centre and radius.

   (c) If $(X, \|\cdot\|)$ is a normed space and $x \in X$, $r > 0$, show that $\bar{B}(x, r)$ is the closure of the open ball $B(x, r)$.

   (d) Show that a normed space $(E, \|\cdot\|)$ is complete if and only if each absolutely convergent series $\sum_{n=1}^{\infty} x_n$ of terms $x_n \in E$ is convergent in E.

2. (a) State the axiom of choice and give a definition of any terminology involved in the statement.

   (b) Let $f: X \to Y$ be a surjective function. Show that there exists a mapping $g: Y \to X$ such that $f \circ g$ is the identity map on Y.

   Show that this is in fact equivalent to the Axiom of Choice. [Hint: Given a family of sets $\{A_i: i \in I\}$, consider the family of disjoint sets given by $B_i = A_i \times \{i\}$. Let $f: \bigcup_i B_i \to I$ be the function which has the value i on B_i.]

   (c) Show that there exists a discontinuous function $f: \mathbb{R} \to \mathbb{R}$ which satisfies the identity $f(x + y) = f(x) + f(y)$. [Hint: Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and use the fact that $\mathbb{R}$ has a basis over $\mathbb{Q}$ that contains 1. Take $f(1) = 0$ and $f(x) = 0$ on other basis elements.]

3. (a) Define the terms *partial order* and *linear order*.

   (b) State Zorn's lemma and explain the terminology involved in the statement.

   (c) Show that in any inner product space, there exist maximal orthonormal subsets.

4. (a) Define *boundedness* for a linear transformation between normed spaces and show that it is equivalent to continuity and to uniform continuity of the transformation. Define the *operator norm* of a bounded linear operator.

   (b) Define the Banach spaces usually denoted $L^p([0,1])$ $(1 \leq p < \infty)$ and $C([0,1])$.

   (c) Show that the inclusion map $C([0,1]) \to L^p([0,1])$ is continuous, linear, has operator norm 1, but is not surjective.

5. (a) Define what is meant by a *Hilbert space*.

   (b) State and prove Bessel's inequality.

   (c) Outline a proof that every separable infinite dimensional Hilbert space is isometrically isomorphic to ℓ^2.

6. (a) Prove that the 'standard basis' of the Hilbert space ℓ^2 is an orthonormal basis for ℓ^2'.

   (b) Show that the sequence space c_0 cannot be a Hilbert space in the usual supremum norm on c_0. [Hint: parallelogram identity.]

   (c) Show that the sequence space c_0 cannot be a Hilbert space in any norm equivalent to the usual supremum norm. [Hint: Is it reflexive?]

   (d) Show that there is $f \in L^2([0,2\pi])$ with

$$\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{|n|+1} \quad (\forall n \in \mathbb{Z}).$$

7. (a) Define the dual space of a normed space and outline a proof that the dual space of a normed space is always a Banach space.

   (b) State the Hahn-Banach theorem and prove the version for complex scalars using the version for real scalars.

8. (a) Suppose that Z is a Banach space and $X, Y \subset Z$ are two closed subspaces with $X \cap Y = \{0\}$ and $X + Y = Z$ (that is $\{x + y : x \in X, y \in Y\} = Z$). Show that the linear map $T: X \oplus_1 Y \to Z$ given by $T(x, y) = x + y$ is an isomorphism of normed spaces.

   (b) Let H be a Hilbert space and $M \subset H$ a closed (linear) subspace. Show that there is a bounded idempotent linear map $P: H \to H$ with range $P(H) = M$.

9. (a) State the uniform boundedness principle.

   (b) Define what is meant by the Fourier series of $f \in L^1[0, 2\pi]$. Also explain what the Dirichlet kernels are and their relation to the partial sums of the Fourier series of a function $f: [0, 2\pi] \to \mathbb{C}$.

   (c) Outline a proof that there is a continuous $f: [0, 2\pi] \to \mathbb{C}$ with $f(0) = f(2\pi)$ so that the Fourier series of f does not converge at $t = 0$.