2E2 Tutorial sheet 7 Solution

[Wednesday December 6th, 2000]

1. Find the $\mathcal{Z}$ transfer function and impulse response for the difference equation

$$x_{k+2} + 3x_{k+1} - 4x_k = v_k$$

(with zero initial conditions).

Solution: Taking $\mathcal{Z}$ transforms of both sides gives

$$(z^2 + 3z - 4)\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) = \mathcal{Z}\left[(v_k)_{k=0}^{\infty}\right](z)$$

or

$$\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) = \frac{1}{z^2 + 3z - 4}\mathcal{Z}\left[(v_k)_{k=0}^{\infty}\right](z)$$

and the $\mathcal{Z}$ transfer function is

$$Y(z) = \frac{1}{z^2 + 3z - 4}$$

the impulse response if the sequence with $\mathcal{Z}$ transform equal to $Y(z)$ and we can find this via partial fractions.

$$\begin{aligned}
\frac{1}{z^2 + 3z - 4} &= \frac{1}{(z+4)(z-1)} \\
&= \frac{A}{z+4} + \frac{B}{z-1} \\
1 &= A(z-1) + B(z+4) \\
\underline{z = -4:} & \\
1 &= -5A \\
& A = -\frac{1}{5} \\
\underline{z = 1:} & \\
1 &= 5B \\
& B = \frac{1}{5} \\
\frac{1}{z^2 + 3z - 4} &= -\frac{1}{5}\frac{1}{z+4} + \frac{1}{5}\frac{1}{z-1}
\end{aligned}$$

z times this would be the $\mathcal{Z}$ transform of the sequence with k^{th} term

$$-\frac{1}{5}(-4)^k + \frac{1}{5}1^k = -\frac{1}{5}(-4)^k + \frac{1}{5}$$

Without the z factor, we must delay this by one. So the impulse repsonse has k^{th} term

$$x_k = \begin{cases} 0 & \text{for } k = 0 \\ -\frac{1}{5}(-4)^{k-1} + \frac{1}{5} & \text{for } k > 0 \end{cases}$$

2. Find the convolution of the sequence $(1)_{k=0}^{\infty} = (1, 1, 1, 1, \ldots)$ with itself.

   Solution: We could work directly with the definition of convolutions of two sequences. The convolution of sequences $(x_k)_{k=0}^{\infty}$ and $(y_k)_{k=0}^{\infty}$ has k^{th} term

$$\sum_{j=0}^{k} x_j y_{k-j}$$

   and when both $x_j = 1$ and $y_{k-j} = 1$ this turns out to be

$$\sum_{j=0}^{k} 1 = k + 1$$

   So the answer is the sequence $(k+1)_{k=0}^{\infty} = (1, 2, 3, \ldots)$.

   Another possible solution would make use of the fact that the $\mathcal{Z}$ transform of the convolution is the product of the $\mathcal{Z}$ transforms, that is

$$\frac{z}{z-1} \times \frac{z}{z-1} = z \frac{z}{(z-1)^2}$$

   Now $\frac{z}{(z-1)^2}$ is the $\mathcal{Z}$ transform of the sequence $(k)_{k=0}^{\infty}$ and the z factor advances it by one, resulting in the same answer again.

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