2E2 Tutorial sheet 6 Solutions
[Wednesday November 29th, 2000]

1. Find the $\mathcal{Z}$ transform of

$$\left(k \sin \frac{k\pi}{3}\right)_{k=0}^{\infty}$$

Solution: We know from the tables that

$$\mathcal{Z}\left[(\sin k\omega)_{k=0}^{\infty}\right](z) = \frac{z \sin\omega}{z^2 - 2z\cos\omega + 1}$$

and one of the properties of $\mathcal{Z}$ is

$$\mathcal{Z}\left[(kx_k)_{k=0}^{\infty}\right](z) = \left(-z\frac{d}{dz}\right)\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z)$$

so that

$$\begin{aligned}
\mathcal{Z}\left[\left(k \sin \frac{k\pi}{3}\right)_{k=0}^{\infty}\right](z) &= \left(-z\frac{d}{dz}\right)\frac{z\sin\frac{\pi}{3}}{z^2 - 2z\cos\frac{\pi}{3} + 1} \\
&= \left(-z\frac{d}{dz}\right)\frac{(\sqrt{3}/2)z}{z^2 - z + 1} \\
&= (-z)(\sqrt{3}/2)\frac{z^2 - z + 1 - z(2z-1)}{(z^2 - z + 1)^2} \\
&= \left(\frac{-\sqrt{3}z}{2}\right)\frac{z^2 - z + 1 - 2z^2 + z}{(z^2 - z + 1)^2} \\
&= \left(\frac{-\sqrt{3}z}{2}\right)\frac{-z^2 + 1}{(z^2 - z + 1)^2} \\
&= \frac{\sqrt{3}}{2}\frac{z(z^2 - 1)}{(z^2 - z + 1)^2}
\end{aligned}$$

2. Find the sequence with $\mathcal{Z}$ transform

$$\frac{z}{z^2 + 2z + 2}$$

Solution: We use partial fractions on $\frac{1}{z}$ times this. The denominator $z^2 + 2z + 2$ has complex roots

$$z = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1} = -1 \pm i$$

and so (complex) partial fractions for this take the form

$$\begin{aligned}
\frac{1}{z^2+2z+2} &= \frac{A}{z-(-1+i)} + \frac{B}{z-(-1-i)}\\
1 &= A(z-(-1-i)) + B(z-(-1+i))\\
\underline{z=-1+i} &\\
1 &= A(-1+i-(-1-i)) = A(2i)\\
& A = \frac{1}{2i} = -\frac{i}{2}\\
\underline{z=-1-i} &\\
1 &= B(-1-i-(-1+i)) = B(-2i)\\
& B = \frac{1}{-2i} = \frac{i}{2}\\
\frac{1}{z^2+2z+2} &= -\frac{i}{2}\frac{1}{z-(-1+i)} + \frac{i}{2}\frac{1}{z-(-1-i)}\\
\frac{z}{z^2+2z+2} &= -\frac{i}{2}\frac{z}{z-(-1+i)} + \frac{i}{2}\frac{z}{z-(-1-i)}
\end{aligned}$$

Thus the required sequence is $(x_k)_{k=0}^{\infty}$ with k^{th} term

$$x_k = -\frac{i}{2}(-1+i)^k + \frac{i}{2}(-1-i)^k$$

We can express this without complex numbers if we use the polar form of

$$-1 \pm i = \sqrt{2}\left(\frac{-1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i\right) = \sqrt{2}(\cos 3\pi/4 \pm i \sin 3\pi/4) = \sqrt{2}e^{\pm 3i\pi/4}$$

so that

$$x_k = \frac{i}{2}2^{k/2}\left(-e^{3ik\pi/4} + e^{-3ik\pi/4}\right) = \frac{i}{2}2^{k/2}(-2i\sin(3k\pi/4)) = 2^{k/2}\sin(3k\pi/4)$$

Richard M. Timoney