2E2 Tutorial sheet 5 Solutions
[Wednesday November 22nd, 2000]

1. Find the sequence with $\mathcal{Z}$ transform

$$\frac{z^4 + 3z - 1}{z^4}$$

Solution: rewriting this as

$$1 + \frac{3}{z^3} - 1z^4$$

and considering the definition of the $\mathcal{Z}$ transform

$$\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) = \mathcal{Z}[(x_0, x_1, x_2, \ldots](z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}$$

we can see that the required sequence is

$$(1, 0, 0, 3, -1, 0, 0, 0, \ldots)$$

2. Find the $\mathcal{Z}$ transform of the solution to the difference equation

$$x_{k+2} + 2x_{k+1} - 3x_k = 1$$

by applying $\mathcal{Z}$ to both sides and using the advancing theorem.

Solution: Applying $\mathcal{Z}$ to both sides, and using linearity, we get

$$\mathcal{Z}\left[(x_{k+2})_{k=0}^{\infty}\right](z) + 2\mathcal{Z}\left[(x_{k+1})_{k=0}^{\infty}\right](z) - 3\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) = \mathcal{Z}[(1, 1, 1, \ldots)](z)$$

The right hand side is the sequence $(1^k)_{k=0}^{\infty} = (a^k)_{k=0}^{\infty}$ (with $a = 1$) and so has $\mathcal{Z}$ transform $\frac{z}{z-1}$ from the table.

From the advancing theorem, we know

$$\mathcal{Z}\left[(x_{k+1})_{k=0}^{\infty}\right](z) = z\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) - zx_0$$

and

$$\mathcal{Z}\left[(x_{k+2})_{k=0}^{\infty}\right](z) = z\mathcal{Z}\left[(x_{k+1})_{k=0}^{\infty}\right](z) - zx_1 = z^2\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) - z^2x_0 - zx_1$$

Thus we have

$$(z^2 + 2z - 3)\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) - z^2x_0 - zx_1 - 2zx_0 = \frac{z}{z-1}$$

$$\begin{aligned}
(z^2 + 2z - 3)\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) &= \frac{z}{z-1} + z^2x_0 + zx_1 + 2zx_0 \\
\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) &= \frac{1}{z^2 + 2z - 3}\left(\frac{z}{z-1} + z^2x_0 + zx_1 + 2zx_0\right)
\end{aligned}$$

3. Assuming $x_0 = x_1 = 0$, find the solution to the above difference equation.

   Solution: From the above we see that the solution in this case has $\mathcal{Z}$ transform

$$\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) = \frac{z}{(z^2+2z-3)(z-1)}$$

   and to find the sequence with that transform, we can use partial fractions and the tables. It turns out that, because nearly all the right hand sides in the table of $\mathcal{Z}$ transforms have a factor z in the numerator, it is simpler to look for the partial fractions of $1/z$ times this and this has the form

$$\begin{aligned}
\frac{1}{(z^2+2z-3)(z-1)} &= \frac{1}{(z+3)(z-1)(z-1)} \\
&= \frac{1}{(z+3)(z-1)^2} \\
&= \frac{A}{z+3} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \\
1 &= A(z-1)^2 + B(z+3)(z-1) + C(z+3) \\
\underline{z=1:} & \\
1 &= 0+0+4C \\
& \quad C = \frac{1}{4} \\
\underline{z=-3:} & \\
1 &= 16A+0+0 \\
& \quad A = \frac{1}{16} \\
\underline{z=0:} & \\
1 &= A - 3B + 3C \\
1 &= \frac{1}{16} - 3B + \frac{3}{4} = \frac{13}{16} - 3B \\
\frac{3}{16} &= -3B \\
& \quad B = -\frac{1}{16} \\
\frac{1}{(z^2+2z-3)(z-1)} &= \frac{1}{16}\frac{1}{z+3} - \frac{1}{16}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} \\
\mathcal{Z}\left[(x_k)_{k=0}^{\infty}\right](z) &= \frac{1}{16}\frac{z}{z+3} - \frac{1}{16}\frac{z}{z-1} + \frac{1}{4}\frac{z}{(z-1)^2} \\
(x_k)_{k=0}^{\infty} &= \frac{1}{16}((-3)^k)_{k=0}^{\infty} - \frac{1}{16}(1^k)_{k=0}^{\infty} + \frac{1}{4}(k)_{k=0}^{\infty} \\
&= \left(\frac{1}{16}((-3)^k + 4k - 1)\right)_{k=0}^{\infty}
\end{aligned}$$

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