2E2 Tutorial sheet 4 Solutions

[Wednesday November 15th, 2000]

1. Find the impulse response for the linear system with response $x(t)$ to an input $g(t)$ given by the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = g(t)$$

($t \geq 0$, zero initial conditions).

Solution: The transfer function for this system is

$$Y(s) = \frac{1}{s^2 + 5s + 6}$$

and the impulse response is the same as the weight function $W(t)$, which is the function with Laplace transform

$$\mathcal{L}[W](s) = Y(s) = \frac{1}{s^2 + 5s + 6}.$$

We can find W by using a partial fractions expansion of Y

$$\begin{aligned}
\frac{1}{s^2+5s+6} &= \frac{1}{(s+2)(s+3)} \\
&= \frac{A}{s+2} + \frac{B}{s+3} \\
1 &= A(s+3) + B(s+2) \\
\underline{s = -2:} & \\
1 &= A \\
\underline{s = -3:} & \\
1 &= B(-1) \qquad B = -1 \\
Y(s) &= \frac{1}{s+2} - \frac{1}{s+3} \\
W(t) &= e^{-2t} - e^{-3t}
\end{aligned}$$

2. Find the Laplace transform of the saw tooth function

$$f(t) = \begin{cases} t & 0 \leq t < c \\ 0 & t \geq c \end{cases} \qquad (c > 0)$$

Solution: From the definition of the Laplace transform

$$\begin{aligned}
\mathcal{L}[f](s) &= \int_0^\infty f(t)e^{-st}\,dt \\
&= \int_0^c f(t)e^{-st}\,dt \\
&\quad \text{since } f(t) = 0 \text{ for } t > c \\
&= \int_0^c te^{-st}\,dt \\
&\qquad \text{Use integration by parts with} \\
&\qquad u = t, \quad dv = e^{-st}\,dt \\
&\qquad du = dt, \quad v = -\frac{1}{s}e^{-st} \\
&= [uv]_{t=0}^c - \int_{t=0}^c v\,du \\
&= \left[-\frac{t}{s}e^{-st}\right]_{t=0}^c + \int_{t=0}^c \frac{1}{s}e^{-st}\,dt \\
&= -\frac{c}{s}e^{-sc} + 0 + \left[\frac{1}{-s^2}e^{-st}\right]_{t=0}^{t=c} \\
&= -\frac{c}{s}e^{-sc} + \left(-\frac{1}{s^2}e^{-sc} + \frac{1}{s^2}\right) \\
&= \frac{1}{s^2} - \frac{c}{s}e^{-sc} - \frac{1}{s^2}e^{-sc}
\end{aligned}$$

3. Find the Laplace transform of the periodic saw tooth function with period $c > 0$ given by

$$\begin{aligned}
f(t) &= t \qquad 0 \le t < c \\
f(t+c) &= f(t)
\end{aligned}$$

Solution: We know the Laplace transform of this is

$$\mathcal{L}[f](s) = \frac{1}{1-e^{-cs}} \int_0^c f(t)e^{-cs}\,dt$$

and this integral is identical to the one we just did in the previous question. So the answer for the periodic saw tooth is

$$\frac{1}{1-e^{-cs}}\left(\frac{1}{s^2} - \frac{c}{s}e^{-sc} - \frac{1}{s^2}e^{-sc}\right)$$

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