2E2 Tutorial sheet 3 Solutions

[Wednesday November 8th, 2000]

1. Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ $(t \geq 0)$.

   Solution: From the definition of convolutions

$$
\begin{aligned}
(f * g)(t) &= \int_0^t f(u)g(t-u)\,du \\
&= \int_0^t ue^{2(t-u)}\,du \\
&= \int_0^t ue^{2t}e^{-2u}\,du \\
&= e^{2t}\int_0^t ue^{-2u}\,du \\
&\quad \text{Use integration by parts with} \\
&\quad U = u, \quad dV = e^{-2u}\,du \\
&\quad dU = du, \quad V = -\frac{1}{2}e^{-2u} \\
&= e^{2t}\int_0^t U\,dV \\
&= e^{2t}\left([UV]_0^t - \int_0^t V\,dU\right) \\
&= e^{2t}\left(\left[-\frac{u}{2}e^{-2u}\right]_0^t - \int_0^t -\frac{1}{2}e^{-2u}\,du\right) \\
&= e^{2t}\left(-\frac{t}{2}e^{-2t} + 0 + \frac{1}{2}\int_0^t e^{-2u}\,du\right) \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left[-\frac{1}{2}e^{-2u}\right]_0^t \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) \\
&= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}
\end{aligned}
$$

2. Use the convolution theorem to find the function $f(t)$ $(t \geq 0)$ with

$$\mathcal{L}[f](s) = \frac{1}{s^2(s-4)}.$$

   Solution: We know $\mathcal{L}[t](s) = \frac{1}{s^2}$ and $\mathcal{L}[e^{4t}](s) = \frac{1}{s-4}$. From the convolution theorem, we see

$$\mathcal{L}[f](s) = \frac{1}{s^2(s-4)} = \mathcal{L}[t](s)\mathcal{L}[e^{4t}](s) = \mathcal{L}[t * e^{4t}](s)$$

so that $f(t)$ is the convolution $t * e^{4t}$.

$$\begin{aligned}
f(t) &= \int_0^t ue^{4(t-u)}\,du \\
&= \int_0^t ue^{4t}e^{-4u}\,du \\
&= e^{4t}\int_0^t ue^{-4u}\,du \\
&\quad \text{Use integration by parts with} \\
&\quad U = u, \quad dV = e^{-4u}\,du \\
&\quad dU = du, \quad V = -\frac{1}{4}e^{-4u} \\
&= e^{4t}\int_0^t U\,dV \\
&= e^{4t}\left([UV]_0^t - \int_0^t V\,dU\right) \\
&= e^{4t}\left(\left[-\frac{u}{4}e^{-4u}\right]_0^t - \int_0^t -\frac{1}{4}e^{-4u}\,du\right) \\
&= e^{4t}\left(-\frac{t}{4}e^{-4t} + 0 + \frac{1}{4}\int_0^t e^{-4u}\,du\right) \\
&= -\frac{t}{4} + \frac{e^{4t}}{4}\left[-\frac{1}{2}e^{-4u}\right]_0^t \\
&= -\frac{t}{4} + \frac{e^{4t}}{4}\left(-\frac{1}{4}e^{-4t} + \frac{1}{4}\right) \\
&= -\frac{t}{4} - \frac{1}{16} + \frac{1}{16}e^{4t}
\end{aligned}$$

3. Use the convolution theorem to find the function $g(t)$ $(t \geq 0)$ with

$$\mathcal{L}[g](s) = \frac{1}{(s-2)^2(s+3)^2}.$$

Solution: We know $\mathcal{L}[te^{2t}](s) = \frac{1}{(s-2)^2}$ and $\mathcal{L}[te^{-3t}](s) = \frac{1}{(s+3)^2}$. From the convolution theorem, we see

$$\mathcal{L}[g](s) = \frac{1}{(s-2)^2(s+3)^2} = \mathcal{L}[te^{2t}](s)\mathcal{L}[te^{-3t}](s) = \mathcal{L}[te^{2t} * te^{-3t}](s)$$

so that $g(t)$ is the convolution $te^{2t} * te^{-3t}$.

$$
\begin{aligned}
g(t) &= \int_0^t ue^{2u}(t-u)e^{-3(t-u)}\,du \\
&= \int_0^t (ut-u^2)e^{-3t}e^{-u}\,du \\
&= e^{-3t}\int_0^t (ut-u^2)e^{5u}\,du \\
&\quad \text{Use integration by parts with} \\
&\quad U = ut-u^2, \quad dV = e^{5u}\,du \\
&\quad dU = (t-2u)\,du, \quad V = \frac{1}{5}e^{5u} \\
&= e^{-3t}\int_0^t U\,dV \\
&= e^{-3t}\left([UV]_0^t - \int_0^t V\,dU\right) \\
&= e^{-3t}\left(\left[\frac{1}{5}(ut-u^2)e^{5u}\right]_0^t - \int_0^t \frac{1}{5}e^{5u}(t-2u)\,du\right) \\
&= e^{-3t}\left(\frac{1}{5}(t^2-t^2)e^{5t} + 0 - \frac{1}{5}\int_0^t e^{5u}(t-2u)\,du\right) \\
&= -\frac{1}{5}e^{-3t}\int_0^t (t-2u)e^{5u}\,du \\
&\quad \text{Use integration by parts again with} \\
&\quad U = t-2u, \quad dV = e^{5u}\,du \\
&\quad dU = -2\,du, \quad V = \frac{1}{5}e^{5u} \\
&= -\frac{1}{5}e^{-3t}\left([UV]_0^t - \int_0^t V\,dU\right) \\
&= -\frac{1}{5}e^{-3t}\left(\left[(t-2u)(\frac{1}{5}e^{5u})\right]_0^t - \int_0^t \frac{1}{5}e^{5u}(-2)\,du\right) \\
&= -\frac{1}{5}e^{-3t}\left((-t)(\frac{1}{5}e^{5t}) - t(\frac{1}{5}1) + \left[\frac{2}{5}e^{5u}\right]_0^t\right) \\
&= \frac{1}{25}te^{2t} + \frac{t}{25}te^{-3t} - \frac{2}{25}e^{-3t}\left(e^{5t}-1\right) \\
&= \frac{1}{25}te^{2t} + \frac{t}{25}te^{-3t} - \frac{2}{125}e^{2t} + \frac{t}{125}e^{-3t}
\end{aligned}
$$

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