2E2 Tutorial sheet 2 Solutions

[Wednesday November 1st, 2000]

1. Use Laplace transform methods and complex numbers to solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 1,$$

with initial conditions $x(0) = \frac{dx}{dt}(0) = 0$.

Solution: From the last tutorial, we know that taking Laplace transforms of both sides leads to

$$\mathcal{L}[x](s) = \frac{1}{s(s^2+2s+2)}$$

and we can factor the denominator using the roots $s = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2} = -1 \pm \sqrt{-1} = -1 \pm i$ of the quadratic. So complex partial fractions take the form

$$\begin{aligned}
\mathcal{L}[x](s) &= \frac{1}{s(s-(-1+i))(s-(-1-i))} \\
&= \frac{A}{s} + \frac{B}{s+1-i} + \frac{C}{s+1+i} \\
1 &= A(s+1-i)(s+1+i) + Bs(s+1+i) + Cs(s+1-i) \\
&= A(s^2+2s+2) + Bs(s+1+i) + Cs(s+1-i) \\
\underline{s = 0:} & \\
1 &= 2A \\
A &= \frac{1}{2} \\
\underline{s = -1+i:} & \\
1 &= 0 + B(-1+i)(2i) + 0 = 2B(-i-1) = -2(1+i)B \\
B &= \frac{1}{-2(1+i)} = -\frac{1-i}{2(1+i)(1-i)} \\
&= -\frac{1-i}{4} = -\frac{1}{4} + \frac{i}{4} \\
\underline{s = -1-i:} & \\
1 &= 0 + 0 + C(-1-i)(-2i) + 0 = 2C(i-1) = 2(-1+i)C \\
C &= \frac{1}{2(-1+i)} = \frac{-1-i}{2(-1+i)(-1-i)} \\
&= -\frac{1+i}{4} = -\frac{1}{4} - \frac{i}{4} \\
&= \overline{B} \\
\mathcal{L}[x](s) &= \frac{1}{2}\frac{1}{s} + \left(-\frac{1}{4} + \frac{i}{4}\right)\frac{1}{s+1-i} + \left(-\frac{1}{4} - \frac{i}{4}\right)\frac{1}{s+1+i}
\end{aligned}$$

Via the tables, we have

$$\begin{aligned}
x(t) &= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right) e^{(-1+i)t} + \left(-\frac{1}{4} - \frac{i}{4}\right) e^{(-1-i)t} \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right) e^{-t}e^{-it} + \left(-\frac{1}{4} - \frac{i}{4}\right) e^{-t}e^{-it} \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{i}{4}\right) e^{-t}(\cos t + i \sin t) + \left(-\frac{1}{4} - \frac{i}{4}\right) e^{-t}(\cos t - i \sin t) \\
&= \frac{1}{2} + \frac{e^{-t}}{4} (-\cos t - \sin t - i \sin t + i \cos t) + \frac{e^{-t}}{4} (-\cos t - \sin t + i \sin t - i \cos t) \\
&= \frac{1}{2} - \frac{e^{-t}}{2} (\cos t + \sin t)
\end{aligned}$$

2. Use Laplace transform methods to solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = \begin{cases} 1, & 0 \leq t < c \\ 0, & t \geq c \end{cases}$$

subject to the initial conditions $x(0) = \frac{dx}{dt}(0) = 0$.

Hint: For this we need a theorem we have not dealt with yet:
If $f(t)$ has a Laplace transform, then $\mathcal{L}\left[H_c(t)f(t-c)\right](s) = e^{-cs}\mathcal{L}[f(t)](s)$. ($H_c$ denotes the Heaviside function with step at c: $H_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$)

Solution: Taking Laplace transforms of both sides and using the tables for the Laplace transform of the right hand side function, leads to

$$\begin{aligned}
(s^2 + 2s - 3)\mathcal{L}[x](s) &= \frac{1 - e^{-cs}}{s} \\
\mathcal{L}[x](s) &= \frac{1 - e^{-cs}}{s(s^2 + 2s - 3)} \\
&= (1 - e^{-cs})\frac{1}{s(s-1)(s+3)} \\
&= (1 - e^{-cs})\left(\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}\right)
\end{aligned}$$

Concentrating on the partial fractions part, we have

$$
\begin{aligned}
\frac{1}{s(s-1)(s+3)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \\
1 &= A(s-1)(s+3) + Bs(s+3) + Cs(s-1) \\
\underline{s = 0:} & \\
1 &= -3A \\
A &= -\frac{1}{3} \\
\underline{s = 1:} & \\
1 &= 0 + 4B + 0 \\
B &= \frac{1}{4} \\
\underline{s = -3:} & \\
1 &= 0 + 0 - 12C \\
C &= -\frac{1}{12}
\end{aligned}
$$

Hence we have

$$
\mathcal{L}[x](s) = (1 - e^{-cs}) \left(-\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} - \frac{1}{12}\frac{1}{s+3} \right)
$$

From the tables, we know that

$$
\mathcal{L}\left[-\frac{1}{3} + \frac{1}{4}e^{-t} - \frac{1}{12}e^{3t} \right](s) = -\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} - \frac{1}{12}\frac{1}{s+3}
$$

and then using the hint we have

$$
x(t) = -\frac{1}{3} + \frac{1}{4}e^{-t} - \frac{1}{12}e^{3t} - H_c(t) \left(-\frac{1}{3} + \frac{1}{4}e^{-(t-c)} - \frac{1}{12}e^{3(t-c)} \right)
$$

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