**2E2 Tutorial sheet 1 Solutions**
[Wednesday October 25th, 2000]

1. Find the Laplace transform of the step function

$$H_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

where $a > 0$.

*Solution:* From the definition of the Laplace transform, we have

$$\mathcal{L}\left[H_a\right](s) = \int_0^\infty H_a(t) e^{-st}\, dt$$

and since $H_a(t) = 0$ for $t < a$, this is

$$\begin{aligned}
\mathcal{L}\left[H_a\right](s) &= \int_a^\infty H_a(t) e^{-st}\, dt \\
&= \int_a^\infty 1 e^{-st}\, dt \\
&= \lim_{b\to\infty} \int_a^b e^{-st}\, dt \\
&\quad \text{(using the definition of an improper integral of this type)} \\
&= \lim_{b\to\infty} \left[\frac{-1}{s} e^{-st}\right]_{t=a}^{t=b} \\
&= \lim_{b\to\infty} \left(\frac{-1}{s} e^{-sb} - \left(\frac{-1}{s} e^{-sa}\right)\right) \\
&= 0 + \frac{1}{s} e^{-sa} \quad \text{(if } s > 0\text{)}
\end{aligned}$$

2. Find the Laplace transform of $1 - H_a(t)$.

*Solution:* We could do this in the same way as the first question, or we could use linearity of the Laplace transform plus the earlier calculation we did that $\mathcal{L}[1](s) = 1/s$ ($s > 0$) and the result of the previous question:

$$\mathcal{L}\left[1 - H_a\right](s) = \mathcal{L}[1](s) - \mathcal{L}\left[H_a\right](s) = \frac{1}{s} - \frac{1}{s} e^{-sa} = \frac{1 - e^{-sa}}{s}$$

(for $s > 0$).

In fact, if we did it directly we could show that

$$\mathcal{L}\left[1 - H_a\right](s) = \int_0^a e^{-st}\, dt = \begin{cases} \frac{1-e^{-sa}}{s} & s \neq 0 \\ a & s = 0. \end{cases}$$

3. Find the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & t \ge b \end{cases}$$

where $0 < a < b$.

*Solution:* Again we could use previous results and linearity, because $f(t) = H_a(t) - H_b(t)$ and so

$$\mathcal{L}[f](s) = \mathcal{L}\left[H_a - H_b\right](s) = \frac{1}{s}e^{-sa} - \frac{1}{s}e^{-sb}$$

(for $s > 0$).

Or, we could apply the definition of the Laplace transform to show directly that

$$\mathcal{L}[f](s) = \int_a^b 1e^{-st}\,dt = \frac{-1}{s}e^{-sb} - \left(\frac{-1}{s}e^{-sa}\right)$$

(as calculated in question 1) without any restriction on $s$ except $s \neq 0$. For $s = 0$ we get the value $b - a$.

4. Find the Laplace transform of both sides of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 1$$

with initial conditions $x(0) = \dfrac{dx}{dt}(0) = 0$.

*Solution:* Using linearity of $\mathcal{L}$, plus the property of Laplace transforms of derivatives (simpler with zero initial conditions), we get

$$\begin{aligned} \mathcal{L}\left[\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x\right](s) &= \mathcal{L}[1](s) \\ \mathcal{L}\left[\frac{d^2x}{dt^2}\right](s) + 2\mathcal{L}\left[\frac{dx}{dt}\right](s) + 2\mathcal{L}[x](s) &= \frac{1}{s} \\ s^2\mathcal{L}[x](s) + 2s\mathcal{L}[x](s) + 2\mathcal{L}[x](s) &= \frac{1}{s} \\ (s^2 + 2s + 2)\mathcal{L}[x](s) &= \frac{1}{s} \end{aligned}$$

Richard M. Timoney