These questions on the Mathematics paper of the Foundation Scholarship Examination for Engineering 2001 were set by R. Timoney.

10 (a) State and prove the formula expressing the Laplace transform $\mathcal{L}[f^{(n)}](s)$ (of the n^{th} derivative of a function $f(t)$ ($t > 0$)) in terms of $\mathcal{L}[f](s)$.

(b) Compute (from the definition) the Laplace transform of the step function

$$H_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

where $a > 0$.

(c) Use Laplace transform methods and complex numbers to solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = \begin{cases} 1, & 0 \leq t < c \\ 0, & t \geq c \end{cases}$$

with initial conditions $x(0) = \dfrac{dx}{dt}(0) = 0$.

11 (a) Define the $\mathcal{Z}$ transform of a sequence $(x_k)_{k=0}^{\infty}$. Define the convolution of two sequences $(x_k)_{k=0}^{\infty}$ and $(y_k)_{k=0}^{\infty}$ and show that the $\mathcal{Z}$ transform of the convolution is the product of the $\mathcal{Z}$ transforms.

(b) Find the $\mathcal{Z}$ transform of

$$\left(\sum_{j=1}^{k} j \sin \frac{(k-j)\pi}{3} \right)_{k=0}^{\infty}$$

(c) Which of the following discrete linear systems are stable (v_k represents the input, x_k the output at step k).

i.

$$2y_{k+2} + 3y_{k+1} + y_k = v_k$$

ii.

$$3y_{k+2} + 9y_{k+1} + 2y_k = v_k$$