2E1 (Timoney) Tutorial sheet 8
[Tutorials November 29 – 30, 2006]

Name: Solutions

1. Show that the function $u(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for $(x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. (This is called a partial differential equation, and this one is one of the most well known, Lapalace's equation.)

Solution:

$$\begin{aligned}
u(x, y, z) &= (x^2 + y^2 + z^2)^{-1/2} \\
\frac{\partial u}{\partial x} &= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) \\
&= -\frac{1}{2}\frac{2x}{(x^2 + y^2 + z^2)^{3/2}} \\
&= \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{-1(x^2 + y^2 + z^2)^{3/2} - (-x)\left(\frac{3}{2}(2x)(x^2 + y^2 + z^2)^{1/2}\right)}{(x^2 + y^2 + z^2)^3} \\
&= \frac{-1(x^2 + y^2 + z^2)^{3/2} + 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}
\end{aligned}$$

We can see then by the symmetry of the function in x, y and z that the calculations of the other second order partials will work out in a similar way and result in

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{-1(x^2 + y^2 + z^2)^{3/2} + 3y^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \\
\frac{\partial^2 u}{\partial z^2} &= \frac{-1(x^2 + y^2 + z^2)^{3/2} + 3z^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}
\end{aligned}$$

Adding up the last 3 formulae we get

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\
&= \frac{-3(x^2 + y^2 + z^2)^{3/2} + 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \\
&= \frac{-3(x^2 + y^2 + z^2)^{3/2} + 3(x^2 + y^2 + z^2)^{3/2}}{(x^2 + y^2 + z^2)^3} \\
&= 0
\end{aligned}$$

You might like to know that Laplace's equation is satisfied by the electrostatic potential coming from a charged object in space, satisfied in the part of space outside the object. The solution $u(x, y, z)$ above would correspond to the potential caused by a unit charge from a point at the origin. As a charged body can be viewed as a lot of tiny charged particles (like say ions or electrons and protons) you can perhaps imagine that there is a way to think of solutions of Laplace's equation in terms of adding up the contributions of solutions like the $u(x, y, z)$ above translated around to represent changes at other points than the origin.

Laplace's equation is also satisfied by steady state temperature distributions $u(x, y, x)$ in a body and by gravitational potentials.

The *Heat equation* is another very basic equation. It would have a time dependeny and would involve (say) $v(x, y, z, t)$ for the heat at position (x, y, z) at time t. It looks like

$$\frac{\partial v}{\partial t} = k^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

where k^2 represents the thermal diffusivity of the material.

One of the other really standard partial differential equations (PDEs) that you will come across in detail in third year is the *wave equation*. It has a second order partial with respect to t in place of the first order $\partial v/\partial t$ and the k^2 will be replaced by something that relates to the tension in whatever is vibrating.

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2. Find the critical points in $\mathbb{R}^2$ of $f(x, y) = 4xy - x^4 - y^4$.

Solution: The idea is to find the points (x, y) where $\nabla f = 0$. So we have to find the solutions of the system

$$\left\{ \begin{array}{rcl} \frac{\partial f}{\partial x} & = & 0 \\ \frac{\partial f}{\partial y} & = & 0 \end{array} \right.$$

$$\left\{ \begin{array}{rcl} 4y - 4x^3 & = & 0 \\ 4x - 4y^3 & = & 0 \end{array} \right.$$

$$\left\{ \begin{array}{rcl} y - x^3 & = & 0 \\ x - y^3 & = & 0 \end{array} \right.$$

$$\left\{ \begin{array}{rcl} y & = & x^3 \\ x & = & y^3 \end{array} \right.$$

It follows that $x = y^3 = (x^3)^3 = x^9$, or $x^9 - x = 0$ or $x(x^8 - 1) = 0$. Thus either $x = 0$ or $x^8 = 1$. If $x^8 = 1$ we have $(x^4)^2 = 1$, thus $x^4 = 1$. That is $(x^2)^2 = 1$, and so $x^2 = 1$, which means $x = 1$ or $x = -1$.

The three cases are then $x = 0$, $x = 1$ and $x = -1$. As we must have $y = x^3$ that gives us 3 critical points $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

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