2E1 (Timoney) Tutorial sheet 6
[Tutorials November 15 – 16, 2006]

Name: Solutions

1. Find the gradient vector evaluated at $(x_0, y_0) = (2, -2)$ for
$$f(x, y) = x^2 e^{xy} - xy^2 + x^4 - y^4$$
Solution: We know $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ and so we work this out.
$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xe^{xy} + x^2 y e^{xy} - y^2 + 4x^3 \\
\frac{\partial f}{\partial y} &= x^3 e^{xy} - 2xy - 4y^3 \\
\frac{\partial f}{\partial x} \mid_{(2,-2)} &= 4e^{-4} - 8e^{-4} - 4 + 32 = -4e^{-4} + 28 \\
\frac{\partial f}{\partial y} \mid_{(2,-2)} &= 8e^{-4} + 8 + 32 = 8e^{-4} + 40 \\
\nabla f \mid_{(2,-2)} &= \left(-4e^{-4} + 28, 8e^{-4} + 40\right)
\end{aligned}$$

2. Find the linear approximation formula for $f(x, y)$ near $(x, y) = (x_0, y_0) = (2, -2)$ (with the same $f(x, y)$ as above).

   Solution: The formula we want is
$$f(x, y) \cong f(x_0, y_0) + \frac{\partial f}{\partial x} \mid_{(x_0,y_0)} (x - x_0) + \frac{\partial f}{\partial y} \mid_{(x_0,y_0)} (y - y_0)$$
and we need $f(x_0, y_0) = f(2, -2) = 4e^{-4} - 8 + 16 - 16 = 4e^{-4} - 8$ together with the values of the partials evaluated at $(2, -2)$ [which we have just found in the previous question as they are the components of the gradient]. So we get
$$f(x, y) \cong \left(4e^{-4} - 8\right) + \left(-4e^{-4} + 28\right)(x - 2) + \left(8e^{-4} + 40\right)(y + 2)$$

Aside: The idea is that this should be good as long as we take (x, y) close to $(x_0, y_0) = (2, -2)$. If you experiment with numerical values you can find $f(2, -2) = -7.92674$, $f(2.1, -2.1) = -9.2074$ (which is quite different) and the linear approximation formula yields $f(2.1, -2.1) \cong -9.14872$ (which is not that far off considering how quickly the function is changing).

$f(2.01, -2.01) = -8.04951$ and linear approximation gives $f(2.01, -2.01) \cong -8.04894$.

If you remember the theory about what a derivative is we should have
$$\lim_{(x,y)\to(2,-2)} \frac{f(x, y) - (f(2, -2) + f_x(2, -2)(x - 2) + f_y(2, -2)(y + 2))}{\sqrt{(x - 2)^2 + (y - 2)^2}} = 0.$$

If you compute the quantity inside the limit for $(x, y) = (2.1, -2.1)$ you get -0.414926 (0.4 is not so close to 0). For $(x, y) = (2.01, -2.01)$ you get -0.040817.

This means that at $(2.01, -2.01)$ we have the expected behaviour that the error in the linear approximation is small compared to the size of the distance of the point away from $(x_0, y_0) = (2, -2)$, but for $(x, y) = (2.1, -2.1)$ the linear approximation is not so dramatically accurate (in this example).

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3. Find the direction $\mathbf{u}$ in which the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ is largest (for the same function $f(x, y)$ as in the previous problem, and the same point (x_0, y_0)).

   Solution: We know, since $D_{\mathbf{u}}f(x_0, y_0) = \nabla f \mid_{(x_0,y_0)} \cdot \mathbf{u}$, that $D_{\mathbf{u}}f(x_0, y_0)$ is as large as it can be only when $\mathbf{u}$ is the unit vector in the same direction as the gradient. Thus $\mathbf{u}$ has to be

$$\mathbf{u} = \frac{1}{\left\|\nabla f \mid_{(2,-2)}\right\|} \nabla f \mid_{(2,-2)} = \frac{1}{\sqrt{(-4e^{-4} + 28)^2 + (8e^{-4} + 40)^2}} \left(-4e^{-4} + 28, 8e^{-4} + 40\right)$$

   If you like you could calculate that the norm is 48.9045 and $\mathbf{u} = (0.571047, 0.820917)$ (approximate numerical values).

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