2E1 (Timoney) Tutorial sheet 1

[Tutorials October 11 – 12, 2006]

Name: Solutions

Consider the vector function (with values in $\mathbb{R}^3$)

$$\mathbf{x}(t) = (\cos t^2, \sin t^2, t^2)$$

1. Show that the image curve lies in the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

and sketch the set in $\mathbb{R}^3$.

Solution: Taking $x = \cos t^2 = \cos(t^2)$, $y = \sin t^2$ and $z = t^2$ with any value of t we find that $x^2 + y^2 = \cos^2 t^2 + \sin^2 t^2 = 1$ always. (Reason: $\cos^2 \theta + \sin^2 \theta = 1$ for all θ and here we are taking $\theta = t^2$).

For the sketch we should get a cylinder of radius 1 centered along the z-axis. A sketch (also showing the curve wrapping around it) is included at the end.

2. Find the derivative $d\mathbf{x}/dt$ and the unit tangent vector $\mathbf{T}$.

Solution:

$$\frac{d\mathbf{x}}{dt} = \left(\frac{d}{dt}\cos t^2, \frac{d}{dt}\sin t^2, \frac{d}{dt}t^2\right) = (-2t\sin t^2, 2t\cos t^2, 2t)$$

The maginitude of this vector is

$$\left\|\frac{d\mathbf{x}}{dt}\right\| = \sqrt{4t^2\sin^2 t^2 + 4t^2\cos^2 t^2 + 4t^2} = \sqrt{4t^2(\sin^2 t^2 + \cos^2 t^2) + 4t^2} = \sqrt{8t^2}.$$

This simplifies to $2\sqrt{2}t$ —- well actually it simplifies to $2\sqrt{2}|t|$ and $2\sqrt{2}t$ is only correct for $t \geq 0$, but we will stick to the $2\sqrt{2}t$ to make life easier.[1] So the unit tangent vector is

$$\mathbf{T} = \frac{\frac{d\mathbf{x}}{dt}}{\left\|\frac{d\mathbf{x}}{dt}\right\|} = \frac{1}{2t\sqrt{2}}(-2t\sin t^2, 2t\cos t^2, 2t) = \left(-\frac{1}{\sqrt{2}}\sin t^2, \frac{1}{\sqrt{2}}\cos t^2, \frac{1}{\sqrt{2}}\right)$$

3. Find the vector $d\mathbf{T}/dt$ and show that it is perpendicular to $\mathbf{T}$.

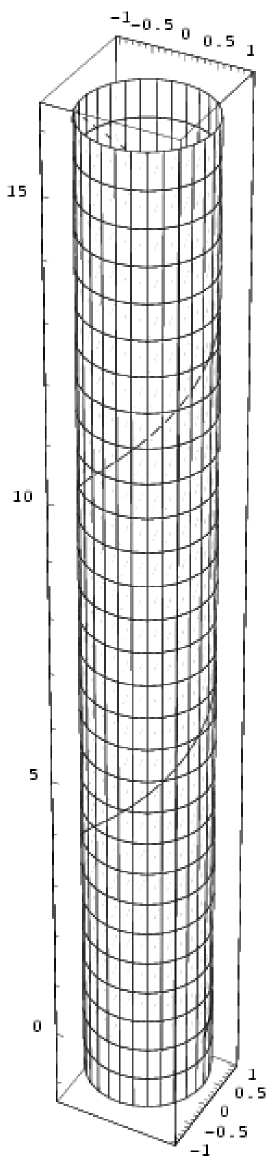
Solution:

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= \left(\frac{d}{dt}\left(-\frac{1}{\sqrt{2}}\sin t^2\right), \frac{d}{dt}\frac{1}{\sqrt{2}}\cos t^2, \frac{d}{dt}\frac{1}{\sqrt{2}}\right) \\ &= (-\sqrt{2}t\cos t^2, -\sqrt{2}t\sin t^2, 0)\end{aligned}$$

[1] For $t < 0$ the answer we get for $\mathbf{T}$ is minus the right answer. For $t = 0$, there is really no answer for $\mathbf{T}$ as we cannot find a unit vector in the direction of the zero vector.

To show the vectors are perpendicular, show that their dot product is 0. To take the dot product we use $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3$ and we find

$$\mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = +t \sin t^2 \cos t^2 - t \cos t^2 \sin t^2 + 0 = 0.$$



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