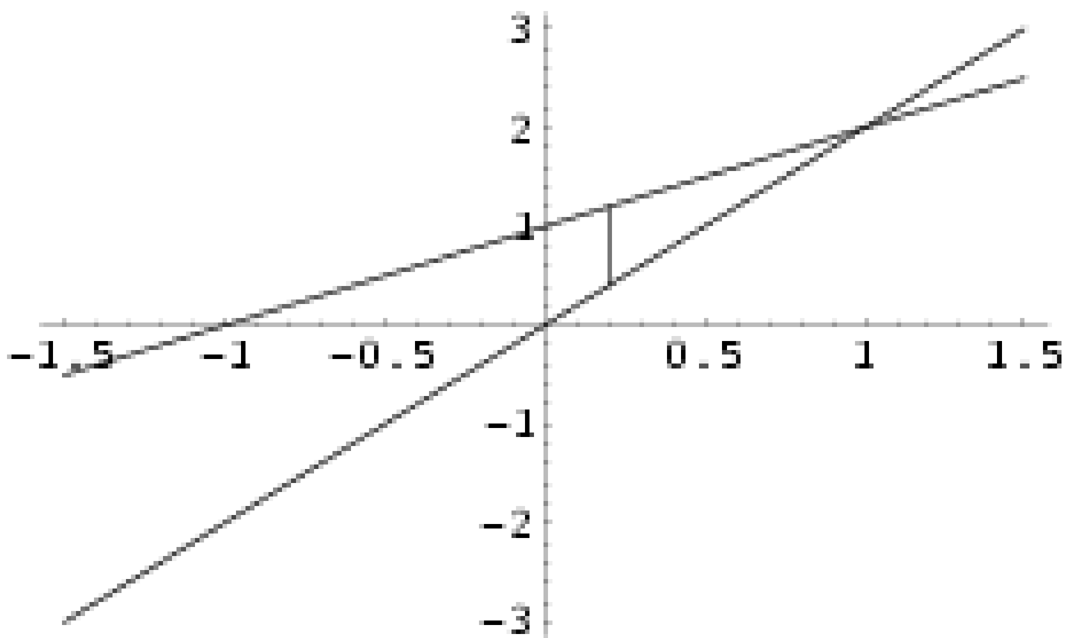
2E1 (Timoney) Tutorial sheet 11
[Tutorials January 17 – 18, 2007]

Name: Solutions

1. Find $\iint_R x^3 + y^2\, dx\, dy$ when R is the region in the plane bounded by the y-axis and the lines $y = 2x$, $y = x + 1$.

   Solution: A picture of R is good to have



In this case it is easier to integrate with respect to y first (keeping x fixed).

$$\begin{aligned}
\iint_R x^3 + y^2\, dx\, dy &= \int_{x=0}^{x=1} \left(\int_{y=2x}^{y=x+1} x^3 + y^2\, dy \right) dx \\
&= \int_{x=0}^{x=1} \left[x^3 y + \frac{y^3}{3} \right]_{y=2x}^{y=x+1} dx \\
&= \int_{x=0}^{x=1} x^3(x+1-2x) + \frac{1}{3}(x+1)^3 - \frac{8}{3}x^3\, dx \\
&= \int_0^1 x^3 - x^4 + \frac{1}{3}(x+1)^3 - \frac{8}{3}x^3\, dx \\
&= \left[\frac{x^4}{4} - \frac{x^5}{5} + \frac{1}{12}(x+1)^4 - \frac{2}{3}x^4 \right]_0^1 \\
&= \frac{1}{4} - \frac{1}{5} + \frac{16}{12} - \frac{2}{3} - \frac{1}{12} \\
&= \frac{19}{30}
\end{aligned}$$

2. Express the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

as an iterated integral (*i.e.* set up the triple integral in terms of single integrals, but do not evaluate it).

Solution: Denoting the ellipsoid by D, its volume is given by

$$\iiint_D 1\,dxdydz.$$

We integrate first with respect to z, keeping (x, y) fixed. The limits for z arise from expressing the equation for the surface of the ellipsoid in the form

$$z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Having integrated with respect to z, we must then take the double integral of the result over the vertical projection (or "shadow") of the ellipsoid onto the xy-plane. This projection is, in this case, the section of the ellipsoid through $z = 0$, or the inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this way we arrive at the answer

$$\int_{x=-a}^{x=a} \left(\int_{y=-b\sqrt{1-x^2/a^2}}^{y=b\sqrt{1-x^2/a^2}} \left(\int_{z=-c\sqrt{1-x^2/a^2-y^2/b^2}}^{z=c\sqrt{1-x^2/a^2-y^2/b^2}} 1\,dz \right) dy \right) dx.$$

We could equally do the integrals in abother order. If we did the indegral dx first, then dy and finally dz we would get

$$\int_{z=-c}^{z=c} \left(\int_{y=-b\sqrt{1-z^2/c^2}}^{y=b\sqrt{1-z^2/c^2}} \left(\int_{x=-a\sqrt{1-y^2/b^2-z^2/c^2}}^{x=a\sqrt{1-y^2/b^2-z^2/c^2}} 1\,dx \right) dy \right) dz.$$

All the orders give integrals that are painful to evaluate.

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