2E1 (Timoney) Tutorial sheet 10
[Tutorials January 10 – 11, 2007]

Name: Solutions

1. Find cylindrical coordinates (r, θ, z) for the point with cartesian coordinates $(1, -1, 3)$.
   Solution: We have $(x, y, z) = (1, -1, 3)$. We know $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and $(x, y) = (r\cos\theta, r\sin\theta)$. So $(1, -1) = (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$. We find $\tan\theta = y/x = -1$ and so θ should be $3\pi/4$ or $7\pi/4$. It works to have $\theta = 7\pi/4$.

   The answer is then $(r, \theta, z) = (\sqrt{2}, 7\pi/4, 3)$.

2. Find cartesian (x, y, z) coordinates for the point with spherical coordinates $(\rho, \theta, \phi) = (4, \pi/3, \pi/4)$.

   Solution: We have $z = \rho\cos\phi = 4\cos\pi/4 = 4/\sqrt{2} = 2\sqrt{2}$, $r = \rho\sin\phi = 4\sin\pi/4 = 2\sqrt{2}$, $x = r\cos\theta = \rho\sin\phi\cos\theta = 2\sqrt{2}\cos(\pi/3) = \sqrt{2}$ and $y = r\sin\theta = \rho\sin\phi\sin\theta = 2\sqrt{2}\sin(\pi/3) = \sqrt{2}\sqrt{3} = \sqrt{6}$.

   The answer is then $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.

3. Find the Taylor expansion up to second order of $f(x, y) = \cos(xy) - x^2 - y^2$ around $(x_0, y_0) = (1, 2)$.

   Solution: Using subscript notation for partials we want to compute

$$
\begin{aligned}
& f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
+ & \frac{1}{2!}\left(f_{xx}(x_0, y_0)(x - x_0)^2 + f_{yy}(x_0, y_0)(y - y_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0)\right)
\end{aligned}
$$

So we compute

$$\begin{aligned}
f(x_0, y_0) &= f(1,2) = \cos 2 - 5 \\
f_x(x, y) &= \frac{\partial f}{\partial x} \\
&= -y \sin(xy) - 2x \\
f_x(x_0, y_0) &= f_x(1,2) = -2\sin 2 - 2 \\
f_y(x, y) &= \frac{\partial f}{\partial y} \\
&= -x \sin(xy) - 2y \\
f_y(x_0, y_0) &= f_y(1,2) = -\sin 2 - 4 \\
f_{xx}(x, y) &= \frac{\partial}{\partial x} f_x(x, y) \\
&= -y^2 \cos(xy) - 2 \\
f_{xx}(x_0, y_0) &= f_{xx}(1,2) = -4\cos 2 - 2 \\
f_{yy}(x, y) &= \frac{\partial}{\partial x} f_y(x, y) \\
&= -x^2 \cos(xy) - 2 \\
f_{yy}(1,2) &= -\cos 2 - 2 \\
f_{xy}(x, y) &= \frac{\partial}{\partial y} f_x(x, y) \\
&= -\sin(xy) - xy\cos(xy) \\
f_{xy}(1,2) &= -\sin 2 - 2\cos 2
\end{aligned}$$

We end up with

$$\begin{aligned}
& (\cos 2 - 5) + (-2\sin 2 - 2)(x-1) + (-\sin 2 - 4)(y-2) \\
+ \quad & \frac{1}{2!}\big((-4\cos 2 - 2)(x-1)^2 + (-\cos 2 - 2)(y-2)^2 + 2(-\sin 2 - 2\cos 2)(x-1)(y-2)\big)
\end{aligned}$$

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