Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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Faculty of Engineering, Mathematics and Science

School of Mathematics

SF Mathematics **Trinity Term 2016**
SF & JS Two Subject Moderatorship

MA2224 — Lebesgue integral (with solutions)

Wednesday, May 18 ? 9.30 — 11.30

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Instructions to Candidates:

Credit will be given for the best THREE questions answered.

All questions have equal weight.

'Formulae & tables' are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.
Apologies for any errors in these solutions!

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) [5 points] If S and T are countably infinite sets, show that $S \times T$ is countable.

Solution: (This was on Tutorial sheet 1 in 2015-16 for $\mathbb{N} \times \mathbb{N}$).

List the elements of S and T as $S = \{s_1, s_2, \ldots\}$, $T = \{t_1, t_2, \ldots\}$ (without repetitions).

We can tabulate the elements of $S \times T$, where the tuples $(s_1, t_1), (s_1, t_2), (s_1, t_3), \ldots$ with first entry s_1 form the first row, those with first entry s_2 the second row and so on.

$$\begin{aligned} S \times T \;=\; \{ \; & (s_1, t_1), (s_1, t_2), (s_1, t_3), (s_1, t_4) \ldots \\ & (s_2, t_1), (s_2, t_2), (s_2, t_3), (s_2, t_4) \ldots \\ & \vdots \\ & \qquad \} \end{aligned}$$

and then list those in a single list by using a zig-zag type pattern

$$S \times T = \{(s_1, t_1), (s_2, t_1), (s_1, t_2), (s_1, t_3), (s_2, t_2), (s_3, t_1), (s_4, t_1), (s_3, t_2), \ldots\}$$

Could $S \times T$ be countable if T is uncountable? (Explain briefly.)

Solution: Yes, if $S = \emptyset$, then $S \times T = \emptyset$ is countable (even if T is uncountable).

(b) [5 points] Let $f \colon A \to B$ be a function and $E \subseteq B$. Show that

$$\chi_{f^{-1}(E)} = \chi_E \circ f.$$

In the case $A = B = \mathbb{R}$, $f(x) = x^2$ and $E = (-3, 0]$, work out both sides of the above equality.

Solution: (The first part was on Tutorial sheet 2 in 2015-16.)

For $a \in A$ we have either $a \in f^{-1}(E)$ or $a \notin f^{-1}(E)$. If $a \in f^{-1}(E)$, then $f(a) \in E$ and so $\chi_E(f(a)) = 1 = \chi_{f^{-1}(E)}(a)$. If $a \notin f^{-1}(E)$, then $f(a) \notin E$ and so $\chi_E(f(a)) = 0 = \chi_{f^{-1}(E)}(a)$.

Thus $\chi_{f^{-1}(E)}(a) = \chi_E(f(a)) = (\chi_E \circ f)(a)$ in both cases.

In the case $f(x) = x^2$, $E = (-3, 0]$

$$f^{-1}(E) = \{x \in \mathbb{R} : f(x) \in E\} = \{x \in \mathbb{R} : x^2 \in (-3, 0]\} = \{x \in \mathbb{R} : x^2 = \{0\}\} = \{0\}$$

So

$$\chi_{f^{-1}(E)}(x) = \chi_{\{0\}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$\chi_E \circ f(x) = \chi_E(f(x)) = \chi_{(-3,0]}(x^2) = \begin{cases} 1 & \text{if } x^2 \in (-3,0] \\ 0 & \text{if } x \notin (-3,0] \end{cases}$$

(c) [5 points] Define what is meant by an algebra of subsets of $\mathbb{R}$ and by a σ-algebra of subsets of $\mathbb{R}$.

Solution: (From the notes)

Definition 1. A collection $\mathcal{A}$ of subsets of $\mathbb{R}$ is called an *algebra of subsets* of $\mathbb{R}$ if

(a) $\emptyset \in \mathcal{A}$

(b) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$

(c) $E_1, E_2 \in \mathcal{A} \Rightarrow E_1 \cup E_2 \in \mathcal{A}$

(In words, $\mathcal{A}$ contains the empty set, is closed under taking complements and under unions (of two members).)

Definition 2. An algebra $\mathcal{A}$ (of subsets of $\mathbb{R}$) is called a σ-algebra ('sigma-algebra') if whenever $E_1, E_2, \ldots \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. [In words: $\mathcal{A}$ is closed under the operation of taking countable unions.]

Fix $E_0 \subseteq \mathbb{R}$ and let $\mathcal{A} = \{E \subset \mathbb{R} : \text{ either } E_0 \subseteq E \text{ or } E \cap E_0 = \emptyset\}$. Show that $\mathcal{A}$ is a σ-algebra of subsets of $\mathbb{R}$.

Solution:

(i) $E = \emptyset \in \mathcal{A}$ since $E \cap E_0 = \emptyset \cap E_0 = \emptyset$

(ii) (Complements) $E \in \mathcal{A}$ implies either

$$E_0 \subseteq E \Rightarrow E^c \cap E_0 = \emptyset \Rightarrow E^c \in \mathcal{A}$$

or

$$E \cap E_0 = \emptyset \Rightarrow E_0 \subseteq E^c \Rightarrow E^c \in \mathcal{A}$$

(iii) (Unions) $E_1, E_2 \in \mathcal{A}$ implies either both $E_1 \cap E_0 = \emptyset$ and $E_2 \cap E_0 = \emptyset$ **or** at least one of $E_0 \subseteq E_1$ and $E_0 \subseteq E_2$.

If $E_1 \cap E_0 = \emptyset$ and $E_2 \cap E_0 = \emptyset$, then

$$(E_1 \cup E_2) \cap E_0 = \emptyset \Rightarrow E_1 \cup E_2 \in \mathcal{A}$$

On the other hand, if at least one of $E_0 \subseteq E_1$ and $E_0 \subseteq E_2$ holds, then

$$E_0 \subseteq E_1 \cup E_2 \Rightarrow E_1 \cup E_2 \in \mathcal{A}$$

(iv) (countable unions) If $E_1, E_2, \ldots \in \mathcal{A}$, then either $E_n \cap E_0 = \emptyset$ for each $n \geq 1$ **or** there is at least one n with $E_0 \subseteq E_n$.

If $E_n \cap E_0 = \emptyset$ for each $n \geq 1$ then

$$\left(\bigcup_{n=1}^{\infty} E_n\right) \cap E_0 = \emptyset \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}.$$

On the other hand, if there is at least one n with $E_0 \subseteq E_n$, then

$$E_0 \subseteq \bigcup_{n=1}^{\infty} E_n \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}.$$

(d) [5 points] For an arbitrary collection $\mathcal{S}$ of subsets of $\mathbb{R}$, show that there is a unique smallest σ-algebra $\Sigma_{\mathcal{S}}$ of subsets of $\mathbb{R}$ with $\mathcal{S} \subseteq \Sigma_{\mathcal{S}}$.

If $\mathcal{S} = \{E_0\}$ contains just one subset $E_0 \subseteq \mathbb{R}$, what is $\Sigma_{\mathcal{S}}$? (Explain succinctly.)

Solution: (First part in notes.)

Proof. This is sort of easy, but in an abstract way. There is certainly one possible σ-algebra containing $\mathcal{S}$, that is $\mathcal{P}(\mathbb{R})$.

What we do is look at all possible σ-algebra, the set

$$\mathscr{S} = \{\Sigma : \Sigma \subseteq \mathcal{P}(\mathbb{R}) \text{ a } \sigma\text{-algebra and } \mathcal{S} \subset \Sigma\}.$$

Then we take their intersection

$$\Sigma_{\mathcal{S}} = \bigcap_{\Sigma \in \mathscr{S}} \Sigma$$

and argue that $\Sigma_{\mathcal{S}}$ is still a σ-algebra. In fact that $\Sigma_{\mathcal{S}} \in \mathscr{S}$ and is the contained in every $\Sigma \in \mathscr{S}$ by the way we defined it.

We check

(i) $\emptyset \in \Sigma_{\mathcal{S}}$

because $\emptyset \in \Sigma$ for each $\Sigma \in \mathscr{S}$ (and $\mathscr{S}$ is not empty — there is at least one Σ in it).

(ii) $E \in \Sigma_{\mathcal{S}} \Rightarrow E^c \in \Sigma_{\mathcal{S}}$

because $E \in \Sigma_{\mathcal{S}} \Rightarrow E \in \Sigma$ for each $\Sigma \in \mathscr{S}$, and so $E^c \in \Sigma$ for each $\Sigma \in \mathscr{S}$. Thus $E^c \in \Sigma_{\mathcal{S}}$.

(iii) $E_1, E_2, \ldots \in \Sigma_{\mathcal{S}} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \Sigma_{\mathcal{S}}$

If $E_n \in \Sigma_{\mathcal{S}}$ for $n = 1, 2, \ldots$, then we have $E_1, E_2, \ldots \in \Sigma$ for each $\Sigma \in \mathscr{S}$. So $\bigcup_{n=1}^{\infty} E_n \in \Sigma$ for each $\Sigma \in \mathscr{S}$, and hence $\bigcup_{n=1}^{\infty} E_n \in \Sigma_{\mathcal{S}}$.

Finally $\mathcal{S} \subset \Sigma_{\mathcal{S}}$ holds because $\mathcal{S} \subset \Sigma$ for each $\Sigma \in \mathscr{S}$. $\square$

If $\mathcal{S} = \{E_0\}$, we could have $E_0 = \emptyset$ or $E_0 = \mathbb{R}$ and in those cases $\Sigma_{\mathcal{S}} = \{\emptyset, \mathbb{R}\}$. Otherwise $\Sigma_{\mathcal{S}} = \{\emptyset, \mathbb{R}, E_0, E_0^c\}$.

(Clearly any σ-algebra that contains E_0 must contain E_0^c too, and all σ-algebras must contain $\emptyset$ and $\emptyset^c = \mathbb{R}$. So we have to have all of $\emptyset, \mathbb{R}, E_0, E_0^c$.

It is easy to see that $\{\emptyset, \mathbb{R}, E_0, E_0^c\}$ contains the empty set and complements. For unions $E_1 \cup E_2$, if $E_1 = E_2$, there is no issue. Also no problem if one is $\emptyset$. If one set is $\mathbb{R}$, so is $E_1 \cup E_2$. Then there remains the case $E_0 \cup E_0^c = \mathbb{R}$. Finite unions follow by induction and infinite unions are actually finite unions as there are only finitely many sets to consider.)

2. (a) [5 points] The 'interval algebra' $\mathscr{I}$ is defined to be those subsets of $\mathbb{R}$ that are finite unions of finite half-open intervals $(a, b]$ and certain other types of sets. What are those other types?

Explain what the standard form of a set in $\mathscr{I}$ is and define the standard length function $m\colon \mathscr{I} \to [0, \infty]$.

Solution: (In the notes.)

The oher types are

$$(-\infty, b], (a, \infty), (-\infty, \infty)$$

(where $a, b \in \mathbb{R}$).

We can write every set in $\mathscr{I}$ uniquely as one of the following

$$\emptyset, \mathbb{R},$$

$$(a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n],$$

$$(-\infty, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n],$$

$$(a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, \infty), \text{ or}$$

$$(-\infty, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, \infty),$$

where

$$a_1 < b_1 < a_2 < b_2 < \cdots a_n < b_n.$$

(This is called the "standard form".)

Definition 1. We define a length function (or 'measure') $m\colon \mathscr{I} \to [0, \infty]$ by taking

$$m((a, b]) = b - a, \qquad m((-\infty, b]) = m((a, \infty)) = m((-\infty, \infty)) = \infty$$

and defining $m(E)$ for general $E \in \mathscr{I}$ as the sum of the lengths of the intervals in the unique representation mentioned above (standard form).

(b) [5 points] Define Lebesgue outer measure $m^*\colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$ and give a direct proof (from the definition) that $m^*(\{x_0\}) = 0$ for each singleton subset $\{x_0\} \subset \mathbb{R}$.

Solution:

Definition 2 (Outer measure)**.** We now define the *outer measure* of an arbitrary subset $S \subseteq \mathbb{R}$ to be

$$m^*(S) = \inf \left\{ \sum_{n=1}^{\infty} m(E_n) : E_1, E_2, \ldots \in \mathscr{I} \text{ with } S \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$

(with the understanding that if $\sum_{n=1}^{\infty} m(E_n) = \infty$ always, then $m^*(S) = \infty$).

Proof. For $S = \{x_0\}$ we can choose $\varepsilon > 0$ and define $E_1 = (x_0 - \varepsilon, x_0]$, $E_n = \emptyset$ for $n > 1$. Then $E_n \in \mathscr{I}$ and $S = \{x_0\} \subset \bigcup_{n=1}^{\infty} E_n = E_1 = (x_0 - \varepsilon, x_0]$, while

$$m^*(S) = m^*(\{x_0\}) \leq \sum_{n=1}^{\infty} m(E_n) = m(E-1) + 0 = \varepsilon.$$

As $\varepsilon > 0$, we must have $m^*(\{x_0\}) = 0$. □

(c) [5 points] Give the definition of a Lebesgue measurable set $F \subset \mathbb{R}$ and show that $F \subset \mathbb{R}$ is Lebesgue measurable if and only if $m^*(S) \geq m^*(S \cap F) + m^*(S \setminus F)$ holds for all $S \subseteq \mathbb{R}$.

Solution: (in the notes)

Definition 3. We say that a subset $F \subset \mathbb{R}$ is *Lebesgue measurable* (or measurable with respect to the outer measure m^*) if for every subset $S \subset \mathbb{R}$,

$$m^*(S) = m^*(S \cap F) + m^*(S \cap F^c).$$

We denote the collection of Lebesgue measurable sets by $\mathscr{L}$.

Proof. If $F \subset \mathbb{R}$ is Lebesgue measurable, then we have equality $m^*(S) = m^*(S \cap F) + m^*(S \cap F^c)$ for each $S \subset \mathbb{R}$, and so we have $\geq$.

By subadditivity of m^*, we always know

$$m^*(S) \leq m^*(S \cap F) + m^*(S \cap F^c).$$

So, conversely, if we assume $m^*(S) \geq m^*(S \cap F) + m^*(S \cap F^c)$, we must have $m^*(S) = m^*(S \cap F) + m^*(S \cap F^c)$ (for each $S \subset \mathbb{R}$). □

(d) [5 points] Prove (from the definition) that the complement of a Lebesgue measurable subset of $\mathbb{R}$ is again Lebesgue measurable.

Show that $\mathbb{Q}$ is a Lebesgue measurable set (in $\mathbb{R}$).

Solution: (First part in the notes.)

Proof. Assume $F \subset \mathbb{R}$ is Lebesgue measurable.

For any $S \subseteq \mathbb{R}$, $m^*(S \cap F^c) + m^*(S \cap (F^c)^c) = m^*(S \cap F^c) + m^*(S \cap F) = m^*(S \cap F) + m^*(S \cap F^c) = m^*(S)$ by measurability of F.

So F^c is Lebesgue measurable. $\square$

We know $m^*(\mathbb{Q}) = 0$ (as $\mathbb{Q}$ is countable and m^* is countably subadditive, we can use the second part of (b) to get this).

So, if $S \subseteq \mathbb{R}$, (using monotonicity of m^*)

$$m^*(S \cap \mathbb{Q}) + m^*(S \cap \mathbb{Q}^c) \leq m^*(\mathbb{Q}) + m^*(S) = 0 + m^*(S) = m^*(S).$$

By (c), $F = \mathbb{Q}$ is Lebesgue measurable.

3. (a) [5 points] Define Lebesgue measurability for a function $f\colon \mathbb{R} \to \mathbb{R}$.

Prove that if $f, g\colon \mathbb{R} \to \mathbb{R}$ are two Lebesgue measurable functions, then their sum $f + g$ must be Lebesgue measurable.

Solution: (in the notes)

Definition 1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *Lebesgue measurable* (or measurable with respect to the measurable space $(\mathbb{R}, \mathscr{L})$) if for each $a \in \mathbb{R}$

$$\{x \in \mathbb{R} : f(x) \le a\} = f^{-1}((-\infty, a]) \in \mathscr{L}.$$

Proof. Fix $a \in \mathbb{R}$ and we aim to show that $\{x \in \mathbb{R} : f(x) + g(x) \le a\} \in \mathscr{L}$. Taking the complement, this would follow from $\{x \in \mathbb{R} : f(x)+g(x) > a\} \in \mathscr{L}$. For any x where $f(x) + g(x) > a$ then $f(x) > a - g(x)$ and there is a rational $q \in \mathbb{Q}$ so that $f(x) > q > a - g(x)$. Then $g(x) > a - q$. So

$$x \in S_q = f^{-1}((q, \infty)) \cap g^{-1}((a - q, \infty))$$

for some $q \in \mathbb{Q}$. On the other hand if $f(x) > q$ and $g(x) > a - q$, then $f(x) + g(x) > a$, from which we conclude that

$$\{x \in \mathbb{R} : f(x) + g(x) > a\} = \bigcup_{q \in \mathbb{Q}} S_q.$$

By the results that f measurable, B Borel implies $f^{-1}(B)$ measurable, (together with the fact that $\mathscr{L}$ is an algebra) $S_q \in \mathscr{L}$ for each q and hence $\{x \in \mathbb{R} : f(x)+g(x) > a\} \in \mathscr{L}$ (because it is countable union of sets $S_q \in \mathscr{L}$). $\square$

(b) [5 points] Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a function, then

$$\Sigma_f = \{E \subseteq \mathbb{R} : f^{-1}(E) \in \mathscr{L}\}$$

is a σ-algebra of subsets of $\mathbb{R}$. ($\mathscr{L}$ denotes the Lebesgue measurable sets.)

Solution: (in the notes)

Proof. That is quite easy to verify.

(i) $f^{-1}(\emptyset) = \emptyset \in \mathscr{L} \Rightarrow \emptyset \in \Sigma_f$

(ii) $E \in \Sigma_f \Rightarrow f^{-1}(E) \in \mathscr{L} \Rightarrow f^{-1}(E^c) = (f^{-1}(E))^c \in \mathscr{L} \Rightarrow E^c \in \Sigma_f$

(iii) $E_1, E_2, \ldots \in \Sigma_f \Rightarrow f^{-1}(E_n) \in \mathscr{L}$ for $n = 1, 2, \ldots$. From this we have $\bigcup_{n=1}^\infty f^{-1}(E_n) \in \mathscr{L}$ and so

$$f^{-1}\left(\bigcup_{n=1}^\infty E_n\right) = \bigcup_{n=1}^\infty f^{-1}(E_n) \in \mathscr{L} \Rightarrow \bigcup_{n=1}^\infty E_n \in \Sigma_f$$

$\square$

(c) [5 points] Explain what a simple function (on $\mathbb{R}$) means, the standard form for such a function, when such a function is Lebesgue measurable, and give the definition of the integral of a non-negative measurable simple function.

Solution: (in the notes)

Definition 2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called a *simple function* if the range $f(\mathbb{R})$ is a finite set.

Proposition 3. *Each simple function $f\colon \mathbb{R} \to \mathbb{R}$ has a representation*

$$f = \sum_{i=1}^n a_i \chi_{E_i}$$

as a linear combination of finitely many characteristic functions of disjoint sets $E_1, E_2, \ldots, E_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Moreover we can assume $a_1, a_2, \ldots, a_n$ are distinct values, that the E_j are all nonempty and that $\bigcup_{j=1}^n E_j = \mathbb{R}$. With these assumptions the representation is called the standard representation *and is unique apart from the order of the sum.*

A simple function is Lebesgue measurable if and only if the sets E_j in the standard representation are all in $\mathscr{L}$.

Definition 4. If $f\colon \mathbb{R} \to [0, \infty)$ is a nonnegative measurable simple function, with standard representation $f = \sum_{j=1}^n a_j \chi_{E_j}$, then we define

$$\int_{\mathbb{R}} f \, d\mu = \sum_{j=1}^n a_j \mu(E_j)$$

(where $\mu(E_j)$ means the Lebesgue measure of E_j). (Note that the integral makes sense in $[0, \infty]$.)

For $E \in \mathscr{L}$, we define

$$\int_E f\, d\mu = \int_{\mathbb{R}} \chi_E f\, d\mu$$

(noting that $\chi_E f$ is also a nonnegative measurable simple function).

(d) [5 points] Explain how the (Lebesgue) integral $\int_{\mathbb{R}} f\, d\mu$ is defined for a nonnegative Lebesgue measurable $f\colon \mathbb{R} \to [0, \infty]$ and give a proof that

$$\int_{\mathbb{R}} \alpha f\, d\mu = \alpha \int_{\mathbb{R}} f\, d\mu$$

holds for such f and $\alpha \geq 0$.

Solution: (from the notes)

Definition 5. For $f\colon \mathbb{R} \to [0, \infty]$ a Lebesgue measurable function, we define

$$\int_{\mathbb{R}} f\, d\mu = \sup \left\{ \int_{\mathbb{R}} s\, d\mu : s\colon \mathbb{R} \to [0, \infty) \text{ a measurable simple function with } s \leq f \right\}$$

Proof. We know there is a monotone increasing sequence $(f_n)_{n=1}^{\infty}$ of measurable simple functions that converges pointwise to f. By the Monotone Convergence Theorem we know that $\int_{\mathbb{R}} f\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu$. But also $(\alpha f_n)_{n=1}^{\infty}$ is a sequence of measurable simple functions that converges pointwise to αf and so we also have $\int_{\mathbb{R}} \alpha f\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \alpha f_n\, d\mu$. Now from the corresponding result for simple (non-negative) functions we have

$$\alpha \int_{\mathbb{R}} f\, d\mu = \lim_{n\to\infty} \alpha \int_{\mathbb{R}} f_n\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \alpha f_n\, d\mu = \int_{\mathbb{R}} \alpha f\, d\mu$$

□

4. (a) [5 points] If $f\colon \mathbb{R} \to [0,\infty)$ is a measurable simple function, prove that $\lambda_f\colon \mathscr{L} \to [0,\infty]$ defined by

$$\lambda_f(E) = \int_{\mathbb{R}} f\chi_E \, d\mu$$

is a measure on $\mathscr{L}$. ($\mathscr{L}$ denotes the Lebesgue measurable sets.)

Solution: (in the notes)

Proof. We know we can write f in the form $f = \sum_{j=1}^n a_j \chi_{E_j}$ with $a_j \geq 0$ and $E_j \in \mathscr{L}$ $(1 \leq j \leq n)$. So

$$\chi_E f = \sum_{j=1}^n a_j \chi_E \chi_{E_j} = \sum_{j=1}^n a_j \chi_{E \cap E_j}$$

and

$$\lambda_f(E) = \int_E f \, d\mu = \int_{\mathbb{R}} \chi_E f \, d\mu = \sum_{j=1}^n a_j \mu(E \cap E_j)$$

We can check easily that $\mu_{E_j}(E) = \mu(E \cap E_j)$ defines a measure on $\mathscr{L}$ and it follows from an earlier result (example) in the notes that $\lambda_f = \sum_{j=1}^n a_j \mu_{E_j}$ is a measure (as a linear combination with nonnegative coefficients of measures). $\square$

(b) [5 points] State and prove the Monotone Convergence Theorem.

Solution: (in the notes)

Theorem 1 (Monotone Convergence Theorem). *If $f_n\colon \mathbb{R} \to [0,\infty]$ is a monotone increasing sequence of (Lebesgue) measurable functions with pointwise limit f, then*

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$$

Proof. Notice that $f(x) = \lim_{n\to\infty} f_n(x) \in [0,\infty]$ is guaranteed to exist as the sequence $(f_n(x))_{n=1}^\infty$ is monotone increasing. Moreover f is measurable and so $\int_{\mathbb{R}} f \, d\mu$ is defined.

From $f_n \leq f_{n+1}$ we have $f_j(x) \leq \lim_{n\to\infty} f_n(x) = f(x)$ for each j. Hence $\int_{\mathbb{R}} f_j \, d\mu \leq \int_{\mathbb{R}} f \, d\mu$ for each j. Also $\int_{\mathbb{R}} f_n \, d\mu \leq \int_{\mathbb{R}} f_{n+1} \, d\mu$ and so the sequence of integrals $\int_{\mathbb{R}} f_n \, d\mu$ is monotone increasing. That means $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$ makes sense in $[0, \infty]$. We have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \sup_{n\geq 1} \int_{\mathbb{R}} f_n \, d\mu \leq \int_{\mathbb{R}} f \, d\mu.$$

It remains to prove the reverse inequality.

Fix a simple function $s \colon \mathbb{R} \to \mathbb{R}$ with $s(x) \leq f(x)$ for all $x \in \mathbb{R}$. We will show $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu \geq \int_{\mathbb{R}} s \, d\mu$ and that will be enough because of the way $\int_{\mathbb{R}} f \, d\mu$ is defined.

To show this, fix $\alpha \in (0, 1)$ and consider

$$E_{n,\alpha} = \{x \in \mathbb{R} : f_n(x) \geq \alpha s(x)\}$$

Notice that

$$\int_{E_{n,\alpha}} \alpha s \, d\mu \leq \int_{E_{n,\alpha}} f_n \, d\mu \leq \int_{\mathbb{R}} f_n \, d\mu \tag{1}$$

Because $f_{n+1} \geq f_n$ we know $E_{n,\alpha} \subset E_{n+1,\alpha}$. For $s(x) = 0$, $x \in E_{n,\alpha}$ and for $s(x) > 0$, $\lim_{n\to\infty} f_n(x) = f(x) \geq s(x) > \alpha s(x)$ implies $x \in E_{n,\alpha}$ when n is large enough. So $\bigcup_{n=1}^{\infty} E_{n,\alpha} = \mathbb{R}$ and a property of the measure $\lambda_{\alpha s}$ tells us that

$$\int_{\mathbb{R}} \alpha s \, d\mu = \lambda_{\alpha s}(\mathbb{R}) = \lambda_{\alpha s}\left(\bigcup_{n=1}^{\infty} E_{n,\alpha} \right) = \lim_{n\to\infty} \lambda_{\alpha s}(E_{n,\alpha}) \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$$

(using (1) at the last step). That gives us

$$\alpha \int_{\mathbb{R}} s \, d\mu = \int_{\mathbb{R}} \alpha s \, d\mu \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu.$$

As that is true for each $\alpha \in (0, 1)$, it follows that $\int_{\mathbb{R}} s \, d\mu \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$, and so

$$\int_{\mathbb{R}} f \, d\mu = \sup\left\{ \int_{\mathbb{R}} s \, d\mu : s \text{ simple measureable, } s \leq f \right\} \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu,$$

completing the proof. □

(c) [5 points] For $f = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} \chi_{[n,n+1)}$, calculate f^+, f^-, show that f is integrable and calculate $\int_{\mathbb{R}} f \, d\mu$ (giving a justification for the calculation).

Solution: (This was on a tutorial sheet, more or less exactly.)

$f^+(x) = \max(f(x), 0)$ is nonzero on the intervals $[n, n+1)$ with n odd. ($f(x) < 0$ on $[n, n+1)$ if n is even and $f(x) = 0$ for $x < 1$.) So

$$f^+(x) = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} \chi_{[2k-1,2k)}$$

For $f^-(x) = \max(-f(x), 0)$, it is nonzero on $[2k, 2k+1)$ ($n = 2k$ even.) So

$$f^-(x) = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \chi_{[2k,2k+1)}$$

By use of the Monotone convergence theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}} f^+ \, d\mu &= \lim_{n\to\infty} \int_{\mathbb{R}} \sum_{k=1}^{n} \frac{1}{2^{2k-1}} \chi_{[2k-1,2k)} \, d\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^{2k-1}} \int_{\mathbb{R}} \chi_{[2k-1,2k)} \, d\mu \\
&= \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} \\
&= \frac{1}{2} \frac{1}{1 - 1/4} = \frac{8}{3}
\end{aligned}$$

By a similar argument again

$$\int_{\mathbb{R}} f^- \, d\mu = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \frac{1}{4} \frac{1}{1 - 1/4} = \frac{1}{3}$$

As f^+ and f^- are both (measurable and) integrable (ie have finite integrals), so is $f = f^+ - f^-$ integrable (the definition of f being integrable) and

$$\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} f^+ \, d\mu - \int_{\mathbb{R}} f^- \, d\mu = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(d) [5 points] State the Lebesgue dominated convergence theorem and its corollary that allows one to establish continuity of functions defined by integrals

$$F(t) = \int_{\mathbb{R}} f(x, t) \, d\mu(x)$$

that depend on a real parameter t.

Solution: (in the notes)

Theorem 2 (Lebesgue dominated convergence theorem)**.** *Suppose* $f_n \colon \mathbb{R} \to [-\infty, \infty]$ *are (Lebesgue) measurable functions such that the pointwise limit* $f(x) = \lim_{n\to\infty} f_n(x)$ *exists. Assume there is an integrable* $g \colon \mathbb{R} \to [0, \infty]$ *with* $|f_n(x)| \le g(x)$ *for each* $x \in \mathbb{R}$*. Then* f *is integrable as is* f_n *for each* n*, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu$$

Theorem 3 (Continuity of integrals)**.** *Assume* $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is such that* $x \mapsto f^{[t]}(x) = f(x, t)$ *is measurable for each* $t \in \mathbb{R}$ *and* $t \mapsto f(x, t)$ *is continuous for each* $x \in \mathbb{R}$*. Assume also that there is an integrable* $g \colon \mathbb{R} \to \mathbb{R}$ *with* $|f(x, t)| \le g(x)$ *for each* $x, t \in \mathbb{R}$*. Then the function* $f^{[t]}$ *is integrable for each* t *and the function* $F \colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$F(t) = \int_{\mathbb{R}} f^{[t]} \, d\mu = \int_{\mathbb{R}} f(x, t) \, d\mu(x)$$

is continuous.