1S3 (Timoney) Tutorial/Exercise sheet 9
[Tutorials April 2 – 13, 2007]

Name: Solutions

Find the following integrals analytically (that is with pencil and paper).

1. $\displaystyle\int \tan^6 x\,dx$

   Solution: Method: Split off two powers and write $\tan^2 x = \sec^2 x - 1$. One part of the integral can be done by substituting $u = \tan x$ and the other is $\int \tan^4 x\,dx$, which can be reduced to $\int \tan^2 x\,dx$ by using the same trick again.

   Or use the reduction formula

$$\begin{aligned}\int \tan^n x\,dx &= \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx\\ \int \tan^6 x\,dx &= \frac{1}{5}\tan^5 x - \int \tan^4 x\,dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \tan^2 x\,dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - \int 1\,dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C\end{aligned}$$

2. $\displaystyle\int \sec^4 x\,dx$

   Solution: We worked out a method that started by splitting off $\sec^2 x$, using integration by parts with $dv = \sec^2 x\,dx$ and $u = \sec^{n-2} x$, then rewriting a resulting $\tan^2 x = \sec^2 x - 1$. We got an equation we could solve to get the reduction formula

$$\begin{aligned}\int \sec^n x\,dx &= \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx\\ \int \sec^4 x\,dx &= \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\int \sec^2 x\,dx\\ &= \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\tan x + C\end{aligned}$$

3. $\displaystyle\int \sqrt{(x+1)^2 - 4}\,dx$

   Solution: Method: Since this involves $\sqrt{u^2 - a^2}$ where $u = x + 1$ and $a = 2$ we can substitute $u = a\cosh t$ or $x + 1 = 2\cosh t$.

We get $dx = 2\sinh t\, dt$

$$\begin{aligned}
\int \sqrt{(x+1)^2 - 4}\, dx &= \int \sqrt{4\cosh^2 t - 4}(2\sinh t)\, dt \\
&= \int \sqrt{4(\cosh^2 t - 1)}(2\sinh t)\, dt \\
&= \int \sqrt{4\sinh^2 t}(2\sinh t)\, dt \\
&= \int 4\sinh^2 t\, dt
\end{aligned}$$

For this we use the formula $\sinh^2 t = \frac{1}{2}(\cosh 2t - 1)$ (analagous to the trig formula $\sin^2\theta = \frac{1}{2}(\cos 2\theta - 1)$) to rewrite the last integral as

$$\begin{aligned}
\int 4\sinh^2 t\, dt &= \int 2(\cosh 2t - 1)\, dt \\
&= \sinh 2t - 2t + C \\
&= 2\sinh t\cosh t - 2t + C
\end{aligned}$$

We have $\cosh t = \frac{x+1}{2}$ and so we mean by this to take $t = \cosh^{-1}\left(\frac{x+1}{2}\right)$ (so $t \geq 0$). From $\cosh^2 t - \sinh^2 t = 1$ we get $\sinh t = \sqrt{\cosh^2 t - 1} = \sqrt{(x+1)^2/4 - 1} = \frac{1}{2}\sqrt{(x+1)^2 - 4}$. Thus our answer simplifies to

$$\begin{aligned}
\int \sqrt{(x+1)^2 - 4}\, dx &= 2\sinh t\cosh t - 2t + C \\
&= \sqrt{(x+1)^2 - 4}\left(\frac{x+1}{2}\right) - 2\cosh^{-1}\left(\frac{x+1}{2}\right) + C
\end{aligned}$$

4. $\displaystyle\int_{-2}^{-1} \frac{x}{x^2 + 4x + 5}\, dx$

Solution: Method: Notice that the denominator can't be factored (has complex roots) and so this is already in partial fraction form.

First rewrite the integral as a part that works by a simple substitution and a part that doesn't.

$$\int_{-2}^{-1} \frac{x}{x^2 + 4x + 5}\, dx = \int_{-2}^{-1} \frac{x+2}{x^2 + 4x + 5}\, dx - \int_{-2}^{-1} \frac{2}{x^2 + 4x + 5}\, dx$$

The first bit works out with $u = x^2 + 4x + 5$, $du = (2x+4)\, dx = 2(x+2)\, dx$. Changing limits to limits for u ($u = 1$ when $x = -2$ and $u = 2$ when $x = -1$) we get

$$\begin{aligned}
\int_{-2}^{-1} \frac{x+2}{x^2+4x+5}\,dx &= \int_{u=1}^{u=2} \frac{x+2}{u}\frac{1}{2(x+2)}\,du \\
&= \int_{u=1}^{u=2} \frac{1}{2u}\,du \\
&= \left[\frac{1}{2}\ln|u|\right]_1^2 \\
&= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 2
\end{aligned}$$

In the other part, complete the square on the denominator $x^2 + 4x + 5 = (x+2)^2 + 1$ and use $x + 2 = \tan\theta$. Change the limits to limits for θ, which are $\theta = 0$ and $\theta = \pi/4$.

We get $dx = \sec^2\theta\,d\theta$ and

$$\begin{aligned}
\int_{-2}^{-1} \frac{2}{x^2+4x+5}\,dx &= \int_{-2}^{-1} \frac{2}{(x+2)^2+1}\,dx \\
&= \int_{\theta=0}^{\theta=\pi/4} \frac{2}{\tan^2\theta+1}\sec^2\theta\,d\theta \\
&= \int_{\theta=0}^{\theta=\pi/4} \frac{2}{\sec^2\theta}\sec^2\theta\,d\theta \\
&= \int_{\theta=0}^{\theta=\pi/4} 2\,d\theta \\
&= [2\theta]_0^{\pi/4} = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

Answer: $\frac{1}{2}\ln 2 - \frac{\pi}{2}$.

5. $\displaystyle\int \frac{x+1}{(x-2)^2(x^2+x+2)}\,dx$

Solution: Method: Partial fractions. Denominator is factored as far as it can go (without using complex roots) because x^2+x+2 has complex roots ($b^2 - 4ac = 1 - 8 < 0$). Partial fractions takes the form

$$\frac{x+1}{(x-2)^2(x^2+x+2)} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{Bx+C}{x^2+x+2}$$

Multiply across by $(x-2)^2(x^2+x+2)$ to get

$$x + 1 = A_1(x-2)(x^2+x+2) + A_2(x^2+x+2) + (Bx+C)(x-2)^2$$

Plugging in $x = 2$ is profitable and gives $A_2 = 3/8$ but there are no other 'magic' values of x. Using $x = 0$, $x = 1$ and $x = -1$ we get a system of 3 equations for A_1, B and C:

$$\begin{aligned} 1 &= -4A_1 + \frac{3}{4} - 2C \\ 2 &= -4A_1 + \frac{3}{2} - (B + C) \\ 0 &= -6A_1 + \frac{3}{4} - 3(-B + C) \end{aligned}$$

The solutions of this system of equations are $A_1 = -7/64$, $B = 7/64$, $C = -3/64$.

For

$$\begin{aligned} \int \frac{Bx + C}{x^2 + x + 2}\, dx &= \frac{1}{64} \int \frac{7x - 3}{x^2 + x + 2}\, dx \\ &= \frac{7}{128} \int \frac{2x + 1}{x^2 + x + 2}\, dx - \frac{13}{128} \int \frac{1}{(x + 1/2)^2 + 7/4}\, dx \end{aligned}$$

use substitution: $u = x^2 + x + 2$ to get $(7/128)\ln(x^2 + x + 2)$ in the first of these integrals and substitute $x + 1/2 = (\sqrt{7}/2)\tan\theta$ in the second part.

Answer: $-\dfrac{7}{64}\ \ln|x-2| - \dfrac{3}{8\ (x-2)} + \dfrac{7}{128}\ \ln(x^2+x+2) - \dfrac{13}{64\sqrt{7}}\ \tan^{-1}\left(\dfrac{2x+1}{\sqrt{7}}\right) + C$

6. $\displaystyle\int \frac{x + 1}{(x + 2)(x + 3)^2}\, dx$

Solution: Method: Partial fractions. Takes the form

$$\frac{x + 1}{(x + 2)(x + 3)^2} = \frac{A_1}{x + 2} + \frac{A_2}{x + 3} + \frac{A_3}{(x + 3)^2}$$

Multiply out to get

$$x + 1 = A_1(x + 3)^2 + A_2(x + 2)(x + 3) + A_3(x + 2)$$

The profitable values of x to use are $x = -2$ (which gives $-1 = A_1$) and $x = -3$ (which gives $-2 = -A_3$ or $A_3 = 2$). Finally $x = 0$ gives $1 = 9A_1 + 6A_2 + 2A_3$ so that $A_2 = 1$.

Answer: $-\ln|x + 2| + \ln|x + 3| - \dfrac{2}{x + 3} + C.$

7. $\displaystyle\int \frac{x^4 + 8x^3 + 21x^2 + 19x + 1}{x^3 + 8x^2 + 21x + 18}\, dx$

Solution: This is a partial fractions problem. As the fraction is not a proper fraction we should begin by using long division of polynomials to get a quotient $Q(x)$ and a remainder $R(x)$.

$$\begin{array}{r|rrrrrrrrr} & x & & & & & & & & \\ \hline x^3 + 8x^2 + 21x + 18 & x^4 & + & 8x^3 & + & 21x^2 & + & 19x & + & 1 \\ & x^4 & + & 8x^3 & + & 21x^2 & + & 18x & & \\ \hline & & & & & & & x & + & 1 \end{array}$$

So $Q(x) = x$ and $R(x) = x + 1$ and we can rewrite

$$\frac{x^4 + 8x^3 + 21x^2 + 19x + 1}{x^3 + 8x^2 + 21x + 18} = Q(x) + \frac{R(x)}{x^3 + 8x^2 + 21x + 18} = x + \frac{x + 1}{x^3 + 8x^2 + 21x + 18}$$

We can integrate x no bother and we concentrate on the (proper) fraction

$$\frac{x + 1}{x^3 + 8x^2 + 21x + 18}$$

(which has numerator of degree $1 < 3 =$ degree of the denominator).

The next step is to completely factorise $x^3 + 8x^2 + 21x + 18$ and it would be a big help if we can find any single root $x = a$ (say) because then by the remainder theorem we know $x - a$ will divide $x^3 + 8x^2 + 21x + 18$ (and leave us with a quadratic to factor). It is a fact that the only 'nice' roots of this will have to be divisors of 18, and that means 1, 2, 3, 6, 9, 18 and minus of all these. Actually none of the positive ones are going to work and so we start with -1, -2 and so on

$$\begin{aligned} x = -1 \quad &: \quad x^3 + 8x^2 + 21x + 18 = -1 + 8 - 21 + 18 = 10 \\ x = -2 \quad &: \quad x^3 + 8x^2 + 21x + 18 = -8 + 32 - 42 + 18 = 0 \end{aligned}$$

Long division by $x + 2$ then must produce no remainder

$$\begin{array}{r|rrrrrrr} & x^2 & + & 6x & + & 9 & & \\ \hline x+2 & x^3 & + & 8x^2 & + & 21x & + & 18 \\ & x^3 & + & 2x^2 & & & & \\ \cline{2-4} & & & 6x^2 & + & 21x & & \\ & & & 6x^2 & + & 12x & & \\ \cline{4-6} & & & & & 9x & + & 18 \\ & & & & & 9x & + & 18 \\ \cline{6-8} & & & & & & & 0 \end{array}$$

So $x^3 + 8x^2 + 21x + 18 = (x + 2)(x^2 + 6x + 9) = (x + 2)(x + 3)^2$.

Fortunately for us (!) we have just worked out the partial fractions and the integral for

$$\frac{x + 1}{(x + 2)(x + 3)^2}$$

in the previous question and so the answer is

$$\begin{aligned} \int \frac{x^4 + 8x^3 + 21x^2 + 19x + 1}{x^3 + 8x^2 + 21x + 18}\, dx &= \int x\, dx + \int \frac{x + 1}{(x + 2)(x + 3)^2}\, dx \\ &= \frac{1}{2}x^2 - \ln|x + 2| + \ln|x + 3| - \frac{2}{x + 3} + C. \end{aligned}$$

8. Find the following (improper) intergral, or show that it does not converge.

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

Solution: Recall **first** that this kind of improper integral is defined as a limit (of integrals that are not improper and so can be worked out)

$$\begin{aligned}
\int_1^\infty \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx \\
&= \lim_{b\to\infty}\left[\frac{1}{1/2}x^{1/2}\right]_1^b \\
&= \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b \\
&= \lim_{b\to\infty} 2\sqrt{b}-2 \\
&= \infty
\end{aligned}$$

This limit is not finite and so we conclude that the improper integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

does *not* converge.

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