1S3 (Timoney) Tutorial sheet 8
[Tutorials February 26 – March 9, 2007]

Name: Solutions

1. An integral $\int_1^3 f(x)\,dx$ is to be calculated based on experimental data for values of $f(x)$. Here is the data:

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
$f(x)$	0.235	0.561	0.862	0.634	0.214	-0.158	-0.517	-0.923	-1.342

Calculate the trapezoidal rule approximation T_1 with only one step, then T_2 with 2 steps (using T_1), T_4 (4 step version) using T_2 and T_8 using T_4 — via the formula

$$T_{2^{n+1}} = \frac{1}{2}T_{2^n} + \frac{b-a}{2^{n+1}}\sum_{j=1}^{2^n} f\left(a + (2j-1)\frac{b-a}{2^{n+1}}\right)$$

Solution:

$$\begin{aligned}
T_1 &= \frac{b-a}{1}\left(\frac{1}{2}f(a) + \frac{1}{2}f(b)\right)\\
&= 0.235 - 1.342 = -1.107\\
T_2 &= \frac{1}{2}T_1 + \frac{3-1}{2}f(2)\\
&= -\frac{1}{2}(1.107) + 0.214 = -0.3395\\
T_4 &= \frac{1}{2}T_2 + \frac{3-1}{4}(f(1.5) + f(2.5))\\
&= -\frac{1}{2}(0.3395) + \frac{1}{2}(0.862 - 0.517) = 0.00275\\
T_8 &= \frac{1}{2}T_4 + \frac{3-1}{8}(f(1.25) + f(1.75) + f(2.25) + f(2.75))\\
&= \frac{1}{2}(0.00275) + \frac{1}{4}(0.561 + 0.634 - 0.158 - 0.923)\\
&= 0.029875
\end{aligned}$$

2. With the same data values as in the previous question find the Simpson's rule approximations S_4 and S_8 to the integral (with 4 and 8 steps). [Hint: $S_{2n} = \frac{4}{3}T_{2n} - \frac{1}{3}T_n$]

Solution:

$$\begin{aligned}
S_4 &= \frac{4}{3}T_4 - \frac{1}{3}T_2\\
&= \frac{4}{3}(0.00275) + \frac{1}{3}(0.3395) = 0.116833\\
S_8 &= \frac{4}{3}T_8 - \frac{1}{3}T_4\\
&= \frac{4}{3}(0.029875) - \frac{1}{3}(0.00275) = 0.0389167
\end{aligned}$$

3. Find $\displaystyle\int \frac{x+1}{(x+2)(x^2+x-2)}\,dx$

   Solution: The method is partial fractions. Finish factoring the denominator of the integrand and then partial fractions takes the form

$$\frac{x+1}{(x-1)(x+2)^2} = \frac{A_1}{x-1} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}$$

   Multiply across by $(x-1)(x+2)^2$ to get

$$x+1 = A_1(x+2)^2 + A_2(x-1)(x+2) + A_3(x-1)$$

   Plugging in $x = 1$ and $x = -2$ is profitable.

$$\begin{aligned}
x = 1: & \\
2 &= A_1 3^2 + A_2(0) + A_3(0) \\
&\Rightarrow A_1 = \frac{2}{9} \\
x = -2: & \\
-1 &= A_1(0) + A_2(0) + A_3(-3) \\
&\Rightarrow A_3 = \frac{1}{3} \\
x = 0: & \\
1 &= 4A_1 - 2A_2 - A_3 \\
&= \frac{8}{9} - 2A_2 - \frac{1}{3} \\
&= \frac{5}{9} - 2A_2 \\
&\Rightarrow 2A_2 = \frac{5}{9} - 1 = -\frac{4}{9} \\
& A_2 = -\frac{2}{9}
\end{aligned}$$

   Notice that plugging in one other value of x (such as $x = 0$, probably the simplest value of x not used before) gives enough information to find A_2.

   Therefore we get

$$\begin{aligned}
\int \frac{x+1}{(x+2)(x^2+x-2)}\,dx &= \int \frac{A_1}{x-1} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}\,dx \\
&= \int \frac{2}{9}\frac{1}{x-1} - \frac{2}{9}\frac{1}{x+2} + \frac{1}{3}\frac{1}{(x+2)^2}\,dx \\
&= \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| - \frac{1}{3(x+2)} + C
\end{aligned}$$

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