**1S3 (Timoney) Tutorial sheet 7**
[Tutorials February 12–23, 2007]

**Name:** Solutions

1. An integral $\int_1^3 f(x)\,dx$ is to be calculated based on experimental data for values of $f(x)$. Here is the data:

| $x$ | 1 | 1.25 | 1.5 | 1.75 | 2 | 2.25 | 2.5 | 2.75 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.235 | 0.561 | 0.862 | 0.634 | 0.214 | $-0.158$ | $-0.517$ | $-0.923$ | $-1.342$ |

Calculate the Riemann sum $\sum_{i=1}^{8} f(x_i^*)(x_i - x_{i-1})$ with $x_i^* = x_i = 1 + 0.25i$.

*Solution:* We have $x_i - x_{i-1} = 0.25$ always and what we want is

$$\begin{aligned}
\sum_{i=1}^{8} f(x_i^*)(x_i - x_{i-1}) &= \sum_{i=1}^{8} f(x_i^*)(0.25) \\
&= 0.25(f(1.25) + f(1.5) + f(1.75) + f(2) \\
&\qquad + f(2.25) + f(2.5) + f(2.75) + f(3)) \\
&= 0.25(0.561 + 0.862 + 0.634 \\
&\qquad + 0.214 - 0.158 - 0.517 - 0.923 - 1.342) \\
&= -0.16725
\end{aligned}$$

2. For the same integral and the same data as in the previous example, find the result of approximating the integral by the Trapezoidal rule formula with $n = 8$ steps.

*Solution:* The formula is

$$T_8 = \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_7 + \frac{1}{2}y_8\right) h$$

where $h = (b-a)/n = (3-1)/8 = 1/4 = 0.25$ and $y_i = f(a + ih) = f(1 + 0.25i)$ $(0 \leq i \leq 8)$ are the values tabulated.

$$\begin{aligned}
T_8 &= \left(\frac{1}{2}f(1) + f(1.25) + f(1.5) + f(1.75) + f(2)\right. \\
&\qquad \left. + f(2.25) + f(2.5) + f(2.75) + \frac{1}{2}f(3)\right)(0.25) \\
&= ((1/2)0.235 + 0.561 + 0.862 + 0.634 \\
&\qquad + 0.214 - 0.158 - 0.517 - 0.923 - (1/2)1.342)(0.25) \\
&= 0.029875
\end{aligned}$$

3. For the same integral and the same data again, find the result of approximating the integral by Simpson's rule formula with $n = 8$ steps.

   *Solution:* The formula is

$$S_8 = \left(\frac{1}{3}y_0 + \frac{4}{3}y_1 + \frac{2}{3}y_2 + \cdots + \frac{4}{3}y_7 + \frac{1}{3}y_8\right)h$$

   where $h = (b-a)/n = (3-1)/8 = 1/4 = 0.25$ and $y_i = f(a+ih) = f(1+0.25i)$ $(0 \leq i \leq 8)$ are again values tabulated.

$$\begin{aligned} S_8 &= \left(\frac{1}{3}f(1) + \frac{4}{3}f(1.25) + \frac{2}{3}f(1.5) + \frac{4}{3}f(1.75) + \frac{2}{3}f(2)\right. \\ &\quad \left. + \frac{4}{3}f(2.25) + \frac{2}{3}f(2.5) + \frac{4}{3}f(2.75) + \frac{1}{3}f(3)\right)(0.25) \\ &= ((1/3)0.235 + (4/3)0.561 + (2/3)0.862 + (4/3)0.634 \\ &\quad +(2/3)0.214 - (4/3)0.158 - (2/3)0.517 - (4/3)0.923 - (1/3)1.342)(0.25) \\ &= 0.0389167 \end{aligned}$$

4. To calculate $\int_0^2 \cos(x^2)\,dx$ via the trapezoidal rule formula $T_n$ with $n$ steps, and to be sure that the value you get is within $0.1$ of the correct value, how large should $n$ be? [Hint: Estimate $M_2 =$ largest value of $|f''(x)|$ for $0 \leq x \leq 2$ via $|f''(x)| = |-4x^2\cos(x^2) - 2\sin(x^2)| \leq 4x^2 + 2 \leq 18$, although $M_2$ is actually smaller.]

   *Solution:* We know the Theorem estimating the difference (for the worst case)

$$\left|\int_0^2 \cos(x^2)\,dx - T_n\right| \leq \frac{b-a}{12}h^2 M_2 \leq \frac{2-0}{12}h^2(18) = 3h^2$$

   Since $h = (b-a)/n = 2/n$, it will be enough to have $n$ big enough so that $3(2/n)^2 < 0.1$. Or $12/n^2 < 0.1$ or $12/0.1 < n^2$. That comes to $n > \sqrt{12/0.1} = \sqrt{120}$. So $n = 11$ will work.

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