1S3 (Timoney) Tutorial sheet 6

[Tutorials January 29 – February 9, 2006]

Name: Solutions

1. For $f(x) = x^3 - 15x$, find the condition number for $f(x)$ at $x = 4$.

   Solution: The condition number is

   $$\frac{xf'(x)}{f(x)} \mid_{x=4} = \frac{x(3x^2 - 15)}{x^3 - 15x} \mid_{x=4} = \frac{4(48 - 15)}{64 - 60} = 33$$

2. Show that there is a solution of $\sin x = -x + 1/2$ in $0 < x < \pi/4$.

   Solution: Rewrite the equation as $\sin x - (-x + 1/2) = 0$ (so it has $= 0$ on one side).

   Consider $f(x) = \sin x - (-x + 1/2) = \sin x + x - 1/2$. This is a continuous function (for all x, and therfore on $[0, \pi/4]$). We have $f(0) = 0 + 0 - 1/2 = -1/2 < 0$ and we have $f(\pi/4) = \sin(\pi/4) + \pi/4 - 1/2 = 1/\sqrt{2} + \pi/4 - 1/2 > 0$ (either because we know $\pi/4 > 3/4 > 1/2$ or because we could use a calculator to work it out as about 0.9925).

   By the Intermediate Value Theorem, we know then that there is some x between $x = 0$ and $x = \pi/4$ where $f(x) = 0$. (What the theorem says is that there has to be at least on such x as the function changes sign from 0 to $\pi/4$.)

   Thus there is a solution, as we are supposed to show.

3. Use the bisection method to locate a solution of the equation $\sin x = x - (1/4)$ in an interval of length 1/8, starting from the fact that there is a solution between 1 and 2.

   Solution: Let $f(x) = \sin x - (x - (1/4)) = \sin x - x + 1/4$ so that the equation is in the form $f(x) = 0$. Since $f(x)$ is continuous and $f(1) = 0.091471 > 0 > f(2) = -0.840703$, there is a solution in $1 < x < 2$ (by the Intermediate Value Theorem).

   So we have a solution in $a < x < b$ with $a = 1$ and $b = 2$. Now look at the midpoint $c = (a + b)/2 = 1.5$ and we see $f(1.5) = -0.25250501$ has opposite sign to $f(1)$. So there is a solution in $1 < x < 1.5$.

   Repeating with $a = 1$, $b = 1.5$, $c = (a + b)/2 = 1.25$ we find $f(1.25) = -0.0510154$ has opposite sign to $f(1)$ and so there is a solution in $1 < x < 1.25$.

   Repeating with $a = 1$, $b = 1.25$, $c = (a + b)/2 = 1.125$ we find $f(1.125) = 0.0272676$ has opposite sign to $f(1.25)$ and so there is a solution in $1.125 < x < 1.25$.

   This interval has length 1/8 and we are finished.

4. Use Newton's method to find a solution of $e^x = 10x$ in the interval $[0, 1]$ to 3 decimal places.

   Solution: We take now $f(x) = e^x - 10x$ so that the equation has the form $f(x) = 0$. As $f(0) = 1 > 0 > f(1) = -7.28172$ we see that the question is reasonable. With Newton's method we start with a guess (say $x_0 = 0.5$) of a solution and get successive guesses by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   Thus we need to know $f'(x) = e^x - 10$ and Newton's method formula comes to

$$x_{n+1} = x_n - \frac{e^{x_n} - 10x_n}{e^{x_n} - 10}$$

   When we work out $x_1, x_2, \ldots$ we get

n	x_n	x_{n+1}
0	0.5	0.0987107
1	0.0987107	0.111822
2	0.111822	0.111833

   Because x_2 and x_3 agree to 3 decimal places we can stop and be fairly confidently we have the solution to 3 decimal places. (In fact x_4 will not be distinguishable from x_3 in the calculator, and the process stabilises.)

Please hand in your work at the end of the hour. Richard M. Timoney