1S3 (Timoney) Tutorial sheet 5

[Tutorials January 15 – January 26, 2007]

1. A differentiable function $f(x)$ (defined for all $x \in \mathbf{R}$) has $f(1) = 3$ and $f(4) = 9$. Show that there must be some x with $f'(x) = 2$. [Hint: Mean Value Theorem]

   Solution: From the mean value theorem we know there is $c \in (a, b)$ with

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   provided f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). In the case of the given f we know it is differentibale everywhere, so continuous everywhere, and thus we are free to use the mean value theorem with aby $a < b$. If we take $a = 1$ and $b = 4$ we find there must be $c \in (1, 4)$ with

   $$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{9 - 3}{3} = 2.$$

   Therefore we see that $x = c$ has $f'(x) = 2$.

2. Find all the asymptotes (horizontal, vertical and/or oblique) for the graph

   $$y = \frac{2x^3 + 11x^2 - 2}{x^2 - 3x + 2}$$

   Solution: For vertical asymptotes, we look at x where the denominator is 0. So we solve $x^2 - 3x + 2 = 0$. Factoring we get $(x - 1)(x - 2) = 0$ and so $x = 1$ and $x = 2$. Checking that the denominator is not also zero at either of these points (for $x = 1$, $2x^3 + 11x^2 - 2 = 11 \neq 0$ and for $x = 2$, $2x^3 + 11x^2 - 2 = 58 \neq 0$) we get $x = 1$ and $x = 2$ as two vertical asymptotes.

   As the numerator has degree (highest power of x) one more than the denominator, we expect an oblique asymptote. Looking at the highest power of x only we get $\frac{2x^3}{x^2} = 2x$ and so we look at

   $$\begin{aligned}
   \lim_{x \to \pm\infty} y - 2x &= \lim_{x \to \pm\infty} \frac{2x^3 + 11x^2 - 2}{x^2 - 3x + 2} - 2x \\
   &= \lim_{x \to \pm\infty} \frac{2x^3 + 11x^2 - 2 - 2x(x^2 - 3x + 2)}{x^2 - 3x + 2} \\
   &= \lim_{x \to \pm\infty} \frac{2x^3 + 11x^2 - 2 - (2x^3 - 6x^2 + 4x)}{x^2 - 3x + 2} \\
   &= \lim_{x \to \pm\infty} \frac{17x^2 - 4x - 2}{x^2 - 3x + 2} \\
   &= \lim_{x \to \pm\infty} \frac{17 - 4/x - 2/x^2}{1 - 3/x + 2/x^2} = 17
   \end{aligned}$$

This means that $\lim_{x\to\infty} y - (2x + 17) = 0$ and the line $y = 2x + 17$ is an (oblique) asymptote.

Since we have an oblique asymptote (close to the graph both at $x \to \infty$ and as $x \to -\infty$), we do not have any horizontal asymptote.

3. Find the linear approximation to $y = \sin x$ near $x = \pi/6$.

   Solution: The linear approximation formula $f(x) \cong f(a) + f'(a)(x - a)$ is to be used with $f(x) = \sin x$ and $a = \pi/6$. We need $f(a) = \sin(\pi/6) = 1/2$, $f'(x) = \cos x$ and $f'(a) = \cos(\pi/6) = \sqrt{3}/2$. Thus

$$\sin x \cong \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$

   is the answer.

4. Find $\sin(\pi/6 + 0.05)$ using a calculator and find the approximate value using the above linear approximation formula.

   Solution: $\sin(\pi/6 + 0.05) = 0.54265836$ is the true value by the calculator. The value from the above approximation is

$$\sin(\pi/6 + 0.05) \cong \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{6} + 0.05 - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}(0.05) = 0.5433017.$$

Please hand in your work at the end of the hour. Richard M. Timoney