1S3 (Timoney) Tutorial sheet 4
[Tutorials December 4–8, 2006, January 8–12, 2007]

1. Show that the hyperbolic functions $\cosh$ and $\sinh$ satisfy the formula

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

(which is an analogue of the trig formula $\sin^2\theta = (1/2)(1 - \cos 2\theta)$ — with different signs).

Solution: We use the definitions $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

This means that

$$\begin{aligned} \sinh^2 x &= (\sinh x)^2 \\ &= \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{(e^x)^2 + (e^{-x})^2 - 2e^x e^{-x}}{4} \\ &= \frac{e^{2x} + e^{-2x} - 2e^{x-x}}{4} \\ &= \frac{e^{2x} + e^{-2x} - 2e^0}{4} \\ &= \frac{e^{2x} + e^{-2x} - 2}{4} \end{aligned}$$

(Here we used the 'laws of exponents' to get $(e^x)^2 = e^{2x}$, $(e^{-x})^2 = e^{-2x}$ and $e^x e^{-x} = e^{x-x} = e^0 = 1$. When you learn more carefully what the exponential function e^x is, you will see that these laws of exponents hold for it.)

Turning now to the right hand side of the equation we want to verify, we find

$$\begin{aligned} \frac{1}{2}(\cosh 2x - 1) &= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2} - 1\right) \\ &= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x} - 2}{2}\right) \\ &= \frac{e^{2x} + e^{-2x} - 2}{4} \end{aligned}$$

This shows that both sides are always equal and establishes that the formula is true.

2. Show that the equation $x + \cosh x = 2$ has a solution (in $\mathbb{R}$). [Hint: Intermediate Value Theorem.]

Solution: Take $f(x) = x + \cosh x - 2$, so that the equation is equivalent to $f(x) = 0$. For the Intermediate Value Theorem, we do need to know that $f(x)$ is continuous on the interval we use in the theorem. But, in fact $f(x)$ is differentiable and so continuous on the whole of $\mathbb{R}$. (Perhaps you have not really learned that yet.)

Now we need an interval where $f(x)$ changes sign. $f(0) = 0 + \cosh 0 - 2 = 0 + (e^0 + e^0)/2 - 2 = (1+1)/2 - 2 = -1 < 0$ and $f(2) = 2 + \cosh 2 - 2 = \cosh 2 > 0$. Thus we can apply the Intermediate Value Theorem on the interval $[a, b] = [0, 2]$ to conclude that there is some $c \in (0, 2)$ with $f(c) = 0$.

3. Show that if $f(x)$ is differentiable on $\mathbb{R}$ and $f(x) = 0$ has 3 different solutions, then $f'(x) = 0$ must have at least 2 different solutions. [Hint: Rolle's theorem.]

   Solution: Say the 3 solutions of $f(x) = 0$ are a_1, a_2 and a_3 and suppose we number them in increasing order, that is so that $a_1 < a_2 < a_3$.

   To see what is going on, it should help to make a little graph of $y = f(x)$ with the 3 points where the graph crosses (or touches) the x-axis at $x = a_1$, $x = a_2$ and $x = a_3$.

   Since $f(a_1) = f(a_2) = 0$, we can apply Rolle's theorem on the interval $[a_1, a_2]$ to conclude there is $c_1 \in (a_1, a_2)$ with $f(c_1) = 0$. (To be sure this is justified we have to check that the hypotheses of the theorem are satisfied. But since $f(x)$ is differentiable on $\mathbb{R}$, it must also be continuous on $\mathbb{R}$. So we can be sure that f is continuous on the closed interval $[a_1, a_2]$ and differentiable on the open interval (a_1, a_2).)

   Since $f(a_2) = f(a_3) = 0$ we can also apply Rolle's theorem on the interval $[a_2, a_3]$ to conclude there is $c_2 \in (a_2, a_3)$ with $f(c_2) = 0$.

   As $c_1 < a_2$ and $a_2 < c_2$, we have $c_1 \neq c_2$ and two solutions of $f(x) = 0$.

4. Where is the graph $y = \dfrac{x-1}{(x-2)^2}$ increasing and where is it decreasing?

   Solution: The idea is to find the intervals where the derivative $\frac{dy}{dx}$ is positive and where it is negative.

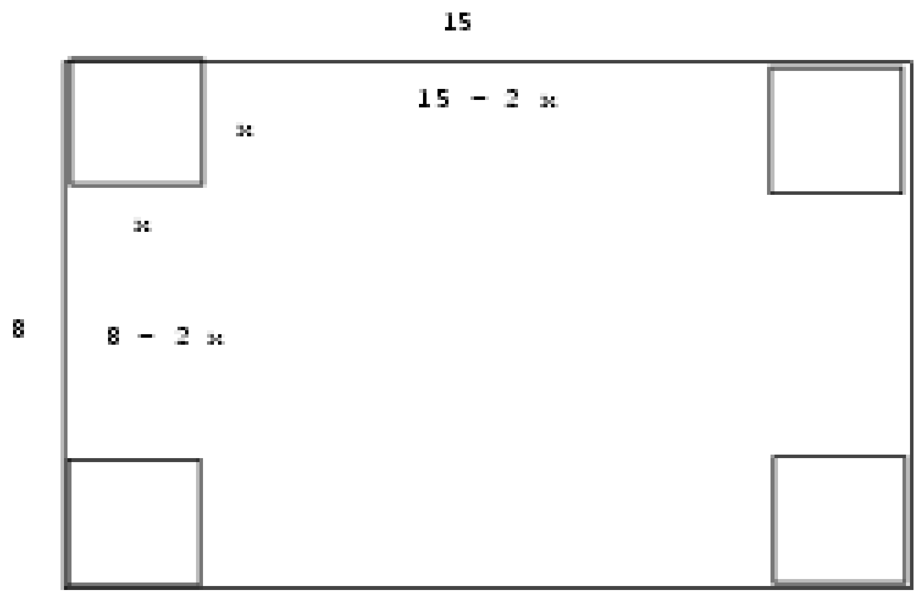
$$\begin{aligned} \frac{dy}{dx} &= \frac{1(x-2)^2 - (x-1)(2)(x-2)}{(x-2)^4} \\ &\qquad \text{(via the quotient rule)} \\ &= \frac{(x-2) - 2(x-1)}{(x-2)^3} \\ &= \frac{-x}{(x-2)^3} \end{aligned}$$

	$x < 0$	$0 < x < 2$	$2 < x$
x	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
$(x-2)^3$	$-$	$-$	$+$
-1	$-$	$-$	$-$
$\frac{dy}{dx} = -\frac{x}{(x-2)^3}$	$-$	$+$	$-$
y	decreasing	increasing	decreasing

From this we see that the graph is increasing for $0 \leq x < 2$ (or on the interval $[0, 2)$). (We cannot include $x = 2$ because the function is not defined there.) It is decreasing for $x \leq 0$ (or on the interval $(-\infty, 0]$) and for $x > 2$ (interval $(2, \infty)$).

5. You are planning to make an open rectangular box from an 8 by 15 cm piece of cardboard by cutting squares from the corners and folding up the sides. What are the dimensions of the box with largest volume you can make this way?

   Solution: Here is a diagram of the situation.



   We have taken x as the length of side of the square to be cut out at each corner.

   The volume of the box is $V = (8 - 2x)(15 - 2x)x$ and the problem is to find the largest value of V that is possible. Clearly $x \geq 0$ is essential and also $x \leq 4$. The end points $x = 0$ and $x = 4$ give $V = 0$ and so the maximum most be at a critical point x in the range $0 < x < 4$. We find $\frac{dV}{dx} = 4(3x^2 - 23x + 30)$ and the solutions of $\frac{dV}{dx} = 0$ are

$$x = \frac{23 \pm \sqrt{23^2 - 360}}{6} = \frac{23 \pm \sqrt{169}}{6} = \frac{23 \pm 13}{6}$$

   or $x = 5/3$ and $x = 6$. Thus the maximum volume happens at $x = 5/3$ and the dimensions required are 14/3, 35/3 and 5/3 cm.