UNIVERSITY OF DUBLIN

MA1S13

TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Natural Sciences **Trinity Term 1999**
JF Human Genetics
JF Computational Physics & Chemistry

Course: 1S Paper 3

Friday, June 11 Luce Hall 14.00 — 17.00

Dr. R.M. Timoney

Brief Solutions/Answers

1. **(a)** Write out the four (base 10) numbers $1, 4, 7, 17$ in binary and explain how a modern computer might use 32 bits to store these numbers.

   Solution: $1_{10} = (1)_2$, $4_{10} = (100)_2$, $7_{10} = (111)_2$ and $(17)_{10} = (10001)_2$.

   Using integer (signed) storage a computer will frequently allocate 32 bits for each integer (including 1 for a possible sign) and these small integers might occupy the 32 slots as follows

1	0	0	...	0	0	0	0	0	1
4	0	0	...	0	0	0	1	0	0
7	0	0	...	0	0	0	1	1	1
17	0	0	...	0	1	0	0	0	1
Bit position:	1	2	...	27	28	29	30	31	32

   (b) Convert the hexadecimal number $(4ab)_{16}$ to octal by first converting it to binary via the "4 binary for one hexadecimal" rule and then using a similar rule to convert to octal.

   Solution: $4 = (100)_2 = (0100)_2$ using a 4-digit number. $a_{16} = 10_{10} = (1010)_2$ and $b_{16} = (1011)_{16}$. Hence

$$(4ab)_{16} = (0100\,1010\,1011)_2 = (10010101011)_2 = (010\,010\,101\,011)_2 = (2253)_8$$

   (c) Convert $23/6$ to binary. Then convert the binary result to scientific notation and give the (binary) mantissa and exponent for it.

Solution: $23/6 = 3 + 5/6 = (11)_2 + 5/6$. Now suppose we write $5/6$ in base 2 and we get digits b_1, b_2, ..., then

$$\begin{aligned}
\frac{5}{6} &= (0.b_1b_2b_3\ldots)_2 \\
& \quad \text{Multiply both sides by 2} \\
\frac{5}{3} &= (b_1.b_2b_3b_4\ldots)_2 \\
\text{Hence } b_1 = 1 & \\
& \quad \text{Take fractional parts of both sides} \\
\frac{2}{3} &= (0.b_2b_3b_4\ldots)_2 \\
& \quad \text{Multiply both sides by 2} \\
\frac{4}{3} &= (b_2.b_3b_4b_5\ldots)_2 \\
\text{Hence } b_2 = 1 & \\
& \quad \text{Take fractional parts of both sides} \\
\frac{1}{3} &= (0.b_3b_4b_5\ldots)_2 \\
& \quad \text{Multiply both sides by 2} \\
\frac{2}{3} &= (b_3.b_4b_5b_6\ldots)_2 \\
\text{Hence } b_3 = 0 & \quad \text{and} \\
\frac{2}{3} &= (0.b_4b_5b_6\ldots)_2
\end{aligned}$$

We see that we are back to a similar form to a few lines back and so $b_4 = b_2$, $b_5 = b_3$, etc. We conclude that

$$\frac{5}{6} = (0.11010101\ldots)_2 = (0.1\overline{10})_2$$

The *binary form* is thus (adding on the whole number part $3 = (11)_2$)

$$\frac{23}{6} = (11.1\overline{10})_2$$

Expressing this in binary *scientific notation*, we get

$$\frac{23}{6} = (1.11\overline{10})_2 \times 2^1$$

The *mantissa* is $(1.11\overline{10})_2$ and the *exponent* is 1.

2. **(a)** Write a `Mathematica` instruction to define a new function $f(x)$ given by

$$f(x) = e^{3x^2+5x}$$

Solution:

```
f[x_] = Exp[3x^2 + 5x]
```

(b) What would you type to ask `Mathematica` to plot the function $f(x)$ just defined in part (a) over the range $-5 \leq x \leq 5$?

Solution:

```
Plot[f[x], {x, -5, 5}]
```

(c) What would you type to ask `Mathematica` to compute the derivative of the function $f(x)$ in part (a)?

Solution:

```
f'[x]
```

(d) Explain the following `Mathematica` programs and how you would use them to find $\int_2^3 f(x)\, dx$ (with $f(x)$ as defined in part (a)).

```
trapdouble[f_, a_, b_, n_, oldt_] :=
  Module [ {h},
        h = N[ (b-a)/(2 n) ];
        oldt/2 + h Sum[ N[ f[a + i h] ],
                    {i, 1, 2 n-1, 2}]
  ];

trap[f_, a_, b_] :=
  Module[ {tn},
        tn = N[ (f[a] + f[b])*(b-a)/2];
        Do [tn = trapdouble[f,a,b,2^k,tn], {k, 0, 10}];
        tn
  ]
```

Solution: Omitted as this relies on Mathematica programming techniques that were not covered in 1999–2000.

3. **(a)** Consider the graph

$$y = \frac{x+4}{x^2-1}$$

Find out where it is increasing, where it is decreasing, its critical points, its local maxima and minima and its asymptotes.

Solution: Apart from the asymptotes the others depend on the sign and the zeros of the first derivative dy/dx. We have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1)(x^2-1) - (x+4)(2x)}{(x^2-1)^2} \\
&= \frac{x^2 - 1 - 2x^2 - 8x}{(x^2-1)^2} \\
&= \frac{-x^2 - 8x - 1}{(x^2-1)^2}
\end{aligned}$$

$$= \frac{-(x^2 + 8x + 1)}{(x^2 - 1)^2}.$$

The roots of $dy/dx = 0$ are the *critical points* and these are

$$x = \frac{-8 \pm \sqrt{64 - 4}}{2} = -4 \pm \sqrt{15}$$

We can write the derivative in factored form as

$$\frac{dy}{dx} = -\frac{(x + 4 - \sqrt{15})(x + 4 + \sqrt{15})}{(x - 1)^2(x + 1)^2}$$

and then we can see that the possible changes of sign are at $-4 \pm \sqrt{15}$ (and that at ± 1 we have bad behaviour — y and its derivative not defined). We can analyse the sign of dy/dx by looking at it on the intervals between these 4 values of x

	$x < -4 - \sqrt{15}$	$-4 - \sqrt{15} < x < -1$	$-1 < x < -4 + \sqrt{15}$
Sign of $x + 4 - \sqrt{15}$	-	-	-
Sign of $x + 4 + \sqrt{15}$	-	+	+
Sign of dy/dx	-	+	+
y	decr.	incr.	incr.

	$-4 + \sqrt{15} < x < 1$	$1 < x$
Sign of $x + 4 - \sqrt{15}$	+	+
Sign of $x + 4 + \sqrt{15}$	+	+
Sign of dy/dx	-	-
y	decr.	decr.

We conclude that the graph is increasing for $-4 - \sqrt{15} < x < -1$ and for $-1 < x < -4 + \sqrt{15}$. It is decreasing on the intervals $x < -4 - \sqrt{15}$, $-4 + \sqrt{15} < x < 1$ and $x > 1$.

We can also see that the critical point $x = -4 - \sqrt{15}$ is a *local minimum* and the critical point $x = -4 + \sqrt{15}$ is a *local maximum*.

The *vertical asymptotes* will occur where the denominator is 0 (and the numerator is not also 0). Thus there are two *vertical asymptotes*: $x = -1$ and $x = 1$.

To find *horizontal asymptotes* we look for

$$\lim_{x \to \pm\infty} y = \lim_{x \to \pm\infty} \frac{x + 4}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{1 + 4/x}{x - 1/x} = 0.$$

Hence the x-axis $y = 0$ is a *horizontal asymptote*.

There are no other asymptotes.

(b) Find the intervals where the graph

$$y = 2x^4 + 4x^3 - 360x^2 + 5x - 7$$

is concave upwards, the intervals where it is concave downwards and its points of inflection.

Solution: The answer relates to the sign of the second derivative d^2y/dx^2 and so we compute that first.

$$\begin{aligned} \frac{dy}{dx} &= 8x^3 - 12x^2 - 720x + 5 \\ \frac{d^2y}{dx^2} &= 24x^2 - 24x - 720 \\ &= 24(x^2 - x - 30) \\ &= 24(x+6)(x-5) \end{aligned}$$

We can see then that the sign of d^2y/dx^2 should change at $x = -6$ and at $x = 5$. Working out the sign of d^2y/dx^2 on the 3 intervals of the real line, we get

	$x < -6$	$-6 < x < 5$	$5 < x$
Sign of $x + 6$	-	+	+
Sign of $x - 5$	-	-	+
Sign of d^2y/dx^2	+	-	+
Concavity of graph	up	down	up

We conclude that the graph is *concave up* on the intervals $x < -6$ and $x > 5$ and that it is *concave down* on the interval $-6 < x < 5$. There are *points of inflection* at $x = -6$ and at $x = 5$ (points on the graph where the concavity changes direction).

4. **(a)** Write down an integral that gives the length of the parametric curve

$$\begin{aligned} x &= 4\cos t \\ y &= 5\sin t \qquad \frac{\pi}{2} \leq t \leq \pi. \end{aligned}$$

Solution: The relevant formula is

$$\text{length} = \int_{\frac{\pi}{2}}^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Thus we compute

$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 5\cos t$$

and we get

$$\text{length} = \int_{\frac{\pi}{2}}^{\pi} \sqrt{16\sin^2 t + 25\cos^2 t}\, dt.$$

This is the required integral.

(b) Write a `Mathematica` command that should find the numerical value of the integral in part (a).

Solution:

```
NIntegrate[ Sqrt[ 16 (Sin[t])^2 + 25 (Cos[t])^2 ], {x, Pi/2, Pi}]
```

(c) Find the total area of the two regions of the plane bounded by the two graphs $y = x^2 - x$ and $y = x^3 - 3x$.

Solution: The formula we want for each of the two regions is

$$\int_a^b (f(x) - g(x))\, dx$$

where $y = f(x)$ is the upper graph and $y = g(x)$ is the lower one, and a and b are the values of x where the graphs intersect.

To find the intersection points, we solve

$$\begin{aligned} x^2 - x &= x^3 - 3x \\ 0 &= x^3 - x^2 - 2x \\ &= x(x^2 - x - 2) \\ &= x(x-2)(x+1) \end{aligned}$$

and we find the points of intersection $x = 0, 2, -1$. Thus the two regions are between $-1 \le x \le 0$ and $0 \le x \le 2$.

Between -1 and 0 we have $x = -1/2$ and there $x^2 - x = 1/4 + 1/2 = 3/4$ while $x^3 - 3x = -1/8 + 3/2 = 11/8 > 3/4$. So $x^3 - 3x$ is the upper graph and the area of the leftmost of the two regions is

$$\begin{aligned} \int_{-1}^0 (x^3 - 3x) - (x^2 - x)\, dx &= \int_{-1}^0 x^3 - x^2 - 2x\, dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_{-1}^0 \\ &= 0 - (\frac{1}{4} + \frac{1}{3} - 1) \\ &= -\frac{1}{4} - \frac{1}{3} + 1 = \frac{-3 - 4 + 12}{12} = \frac{5}{12}. \end{aligned}$$

Between 0 and 2 we have $x = 1$ and there $x^2 - x = 0$ while $x^3 - 3x = -2 < 0$. The area of the

$$\begin{aligned} \int_0^2 (x^2 - x) - (x^3 - 3x)\, dx &= \int_0^2 -x^3 + x^2 + 2x\, dx \\ &= \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 \right]_0^2 \\ &= -4 + \frac{8}{3} + 4 = \frac{8}{3} \end{aligned}$$

Thus the total area of the two regions is $\frac{5}{12} + \frac{8}{3} = \frac{5+32}{12} = \frac{37}{12}$.

5. **(a)** Suppose

$$\int_3^{x^3} f(t)\,dt = 7x^{11} + 8x^6 - 9$$

for some continuous function $f(x)$. Find $f(x)$.

Solution: From the Fundamental Theorem we know that

$$\frac{d}{du}\int_3^u f(t)\,dt = f(u)$$

Using the Chain Rule with $u = x^3$, we get

$$\frac{d}{dx}\int_3^{x^3} f(t)\,dt = \frac{d}{du}\int_3^u f(t)\,dt\frac{du}{dx} = f(u)(3x^2) = 3x^2 f(x^3)$$

As this must be the same as what we get by differentiating $7x^{11} + 8x^6 - 9$, we get

$$3x^2 f(x^3) = 77x^{10} + 48x^5$$

and that leads to

$$f(x^3) = \frac{77x^{10} + 48x^5}{3x^2} = \frac{77x^7 + 48x^3}{3}$$

Replacing x by $x^{1/3}$ we get

$$f(x) = \frac{1}{3}(77x^{7/3} + 48x).$$

(b) Suppose $h(x)$ is a function of $x \in \mathbb{R}$ and that it satisfies both

$$h'(x) = \cos(x^4 + 3x^2 + 11)$$

and $h(2) = 0$. Find a formula for $h(x)$ which involves an integral.

Solution: From the Fundamental Theorem any $h(x)$ of the form

$$h(x) = \int_a^x \cos(t^4 + 3t^2 + 11)\,dt$$

satisfies $h'(x) = \cos(x^4 + 3x^2 + 11)$. If we take $a = 2$ we also have $h(2) = 0$. So the required formula is

$$h(x) = \int_2^x \cos(t^4 + 3t^2 + 11)\,dt$$

6. Find the following integrals analytically (without using numerical approximations or computer algebra):

(a) $\displaystyle\int \sin^4 x \cos^5 x\, dx$

Solution: Since the power of $\cos x$ is odd, we substitute $u = \sin x$, $du = \cos x\, dx$, and we get

$$\begin{aligned}
\int \sin^4 x \cos^5 x\, dx &= \int u^4 \cos^5 x \frac{du}{\cos x} \\
&= \int u^4 \cos^4 x\, du \\
&= \int u^4 (\cos^2 x)^2\, du \\
&= \int u^4 (1 - \sin^2 x)^2\, du \\
&= \int u^4 (1 - u^2)^2\, du \\
&= \int u^4 (1 - 2u^2 + u^4)\, du \\
&= \int (u^4 - 2u^6 + u^8)\, du \\
&= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C
\end{aligned}$$

(b) $\displaystyle\int e^x \cos 2x\, dx$

Solution: We integrate by parts taking $u = e^x$, $dv = \cos 2x\, dx$. Then $du = e^x\, dx$ and $v = \frac{1}{2}\sin 2x$. Hence

$$\begin{aligned}
\int e^x \cos 2x\, dx &= \int u\, dv \\
&= uv - \int v\, du \\
&= e^x \left(\frac{1}{2}\sin 2x\right) - \int \frac{1}{2}\sin 2x e^x\, dx \\
&= \frac{1}{2}e^x \sin 2x - \frac{1}{2}\int e^x \sin 2x\, dx
\end{aligned}$$

Now we integrate by parts again, using this time $U = e^x$ and $dV = \sin 2x\, dx$ so that $dU = e^x\, dx$ and $V = -\frac{1}{2}\cos 2x$. We get

$$\begin{aligned}
\int e^x \cos 2x\, dx &= \frac{1}{2}e^x \sin 2x - \frac{1}{2}\int U\, dV \\
&= \frac{1}{2}e^x \sin 2x - \frac{1}{2}\left(UV - \int V\, dU\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}e^x \sin 2x - \frac{1}{2}UV + \frac{1}{2}\int V\,dU \\
&= \frac{1}{2}e^x \sin 2x - \frac{1}{2}e^x\left(-\frac{1}{2}\cos 2x\right) + \frac{1}{2}\int -\frac{1}{2}\cos 2x e^x\,dx \\
&= \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int \cos 2x e^x\,dx \\
&= \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x\,dx
\end{aligned}$$

We now have an equation to solve for our unknown integral $\int e^x \cos 2x\,dx$ and we get

$$\begin{aligned}
\frac{5}{4}\int e^x \cos 2x\,dx &= \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x \\
\int e^x \cos 2x\,dx &= \frac{2}{5}e^x \sin 2x + \frac{1}{5}e^x \cos 2x
\end{aligned}$$

This answer is missing the $+C$ and so the final answer is

$$\int e^x \cos 2x\,dx = \frac{2}{5}e^x \sin 2x + \frac{1}{5}e^x \cos 2x + C$$

(c) $\displaystyle\int_0^{\pi/4} \tan^6 x\,dx.$

Solution: We use $\tan^2 x = \sec^2 x - 1$ and split the integral so as to be able to use substitution on one part.

$$\begin{aligned}
\int_0^{\pi/4} \tan^6 x\,dx &= \int_0^{\pi/4} \tan^4 x \tan^2 x\,dx \\
&= \int_0^{\pi/4} \tan^4 x(\sec^2 x - 1)\,dx \\
&= \int_0^{\pi/4} \tan^4 x \sec^2 x\,dx - \int_0^{\pi/4} \tan^4 x\,dx \\
&= \text{Put } u = \tan x \qquad du = \sec^2 x\,dx \\
&= \qquad u = 0 \text{ for } x = 0 \text{ and } u = 1 \text{ for } x = \pi/4 \\
&= \int_{u=0}^{1} u^4\,du - \int_0^{\pi/4} \tan^4 x\,dx \\
&= \left[\frac{1}{5}u^5\right]_0^1 - \int_0^{\pi/4} \tan^4 x\,dx \\
&= \frac{1}{5} - \int_0^{\pi/4} \tan^4\ dx \\
&= \frac{1}{5} - \int_0^{\pi/4} \tan^2 x \tan^2 x\,dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5} - \int_0^{\pi/4} \tan^2 x(\sec^2 x - 1)\,dx \\
&= \frac{1}{5} - \int_0^{\pi/4} \tan^2 x \sec^2 x\,dx + \int_0^{\pi/4} \tan^2 x\,dx \\
&= \frac{1}{5} - \int_0^{1} u^2\,du + \int_0^{\pi/4} \sec^2 x - 1\,dx \\
&= \frac{1}{5} - \left[\frac{1}{3}u^3\right]_0^1 + [\tan x - x]_0^{\pi/4} \\
&= \frac{1}{5} - \left(\frac{1}{3} - 0\right) + \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - 0 \\
&= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} \\
&= \frac{13}{15} - \frac{\pi}{4}
\end{aligned}$$

7. Find the following integrals analytically:

(a) $\displaystyle\int \frac{x+1}{(x+2)(x^2+2x+3)}\,dx$

Solution: This is a partial fractions problem. It is a proper fraction (numerator has degree $= 1 < 3 =$ degree of the denominator). Next step is to factor the denominator completely. But x^2+2x+3 has $b^2 - 4ac = 4 - 12 < 0$ and so has complex roots. So it cannot be factored. Thus the partial fraction decomposition takes the form

$$\frac{x+1}{(x+2)(x^2+2x+3)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+3}$$

Multiplying across by $(x+2)(x^2+2x+3)$, we get

$$x + 1 = A(x^2+2x+3) + (Bx+C)(x+2)$$

Substituting $x = -2$ we get

$$-1 = A(4 - 4 + 3) + 0 = 3A$$

and so $A = -1/3$.

Substituting $x = 0$ gives $1 = 3A + 2C = -1 + 2C$. We deduce $2 = 2C$ and $C = 1$.

Finally substituting another value $x = 1$ (simplest value not used so far) we get $2 = A(6) + (B+C)(3) = 6A + 3B + 3C = -2 + 3B + 3 = 3B + 1$. Hence $B = 1/3$.

Summarising, we have

$$\frac{x+1}{(x+2)(x^2+2x+3)} = \frac{-1}{3(x+2)} + \frac{x+3}{3(x^2+2x+3)}$$

Hence

$$\begin{aligned}
\int \frac{x+1}{(x+2)(x^2+2x+3)}\,dx &= -\int \frac{1}{3(x+2)} + \int \frac{x+3}{3(x^2+2x+3)}\,dx \\
&= -\frac{1}{3}\ln|x+2| + \int \frac{x+3}{3(x^2+2x+3)}\,dx
\end{aligned}$$

This would turn out to be an integral we can manage by a substitution $u = x^2+2x+3$, $du = (2x+2)\,dx$ if the numerator was a multiple of $x+1$. So we split up the integral

$$\begin{aligned}
\int \frac{x+3}{3(x^2+2x+3)}\,dx &= \int \frac{x+1}{3(x^2+2x+3)}\,dx + \int \frac{2}{3(x^2+2x+3)}\,dx \\
&= \int \frac{x+1}{3u}\,\frac{du}{2(x+1)} + \int \frac{2}{3(x^2+2x+3)}\,dx \\
&= \int \frac{1}{6u}\,du + \int \frac{2}{3(x^2+2x+3)}\,dx \\
&= \frac{1}{6}\ln|u| + \int \frac{2}{3(x^2+2x+3)}\,dx \\
&= \frac{1}{6}\ln|x^2+2x+3| + \int \frac{2}{3(x^2+2x+3)}\,dx \\
&= \frac{1}{6}\ln(x^2+2x+3) + \int \frac{2}{3(x^2+2x+3)}\,dx
\end{aligned}$$

For this last integral, we complete the square $x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+2$ and substitute $x+1 = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{2}{3(x^2+2x+3)}\,dx &= \int \frac{2}{3((x+1)^2+2)}\,dx \\
&= \int \frac{2}{3(2\tan^2\theta+2)}\sqrt{2}\sec^2\theta\,d\theta \\
&= \int \frac{2\sqrt{2}}{6(\tan^2\theta+1)}\sec^2\theta\,d\theta \\
&= \int \frac{\sqrt{2}}{3\sec^2\theta}\sec^2\theta\,d\theta \\
&= \int \frac{\sqrt{2}}{3}\,d\theta \\
&= \frac{\sqrt{2}}{3}\theta = \frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)
\end{aligned}$$

Combining these various components, we have

$$\int \frac{x+1}{(x+2)(x^2+2x+3)}\,dx = -\frac{1}{3}\ln|x+2| + \frac{1}{6}\ln(x^2+2x+3) + \frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ (an improper integral).

Solution: This is improper because of the bad behaviour of the integrand at $x = 0$ and so we split the integral into two separate ones

$$\int_{-1}^{0} \frac{1}{x^2}\,dx \text{ and } \int_{0}^{1} \frac{1}{x^2}\,dx$$

If both exist (as finite values), then the whole answer is the sum of the two values. If one fails to converge, then the answer to the problem is that the integral does not converge. For the leftmost integral, we have, by definition

$$\begin{aligned}\int_{-1}^{0} \frac{1}{x^2}\,dx &= \lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x^2}\,dx \\ &= \lim_{b \to 0^-} \left[-\frac{1}{x}\right]_{-1}^{b} \\ &= \lim_{b \to 0^-} \left(-\frac{1}{b} + 1\right) \\ &= \infty\end{aligned}$$

Thus the answer is that the integral diverges (or does not converge).

8. **(a)** A spring with spring constant 12 is fixed at one end and pushes an object attached to its other end from a position where the spring is compressed by 0.7 length units to the position where the spring is in equilibrium. How much work was done by the spring?

Solution: Note: This application of integration was omitted from the course in 1999–2000.

If we imagine the spring positioned on an axis so that the fixed end of the spring is to the left of the origin and the free end has equilibrium position at $x = 0$, then Hook's Law tells us that the Force exerted by the spring when it is extended/compressed to position x is $F(x) = -12x$. Then we are concerned with the work done the object moves from $x = -0.7$ (spring compressed by 0.7) to $x = 0$ (equilibrium). The formula is

$$\begin{aligned}\text{Work} &= \int_{-0.7}^{0} F(x)\,dx \\ &= \int_{-0.7}^{0} -12x\,dx \\ &= [-6x^2]_{-0.7}^{0} \\ &= 0 - (-6(-0.7)^2) = 6(0.49) = 2.94\end{aligned}$$

(b) A hole of radius 3 (units of length) is drilled vertically down through the centre of a solid hemisphere of radius 5 when the hemisphere has its flat face horizontal. What is the volume of the remaining object?

Solution: We can approach this by looking at the top half of the object as the volume swept out by rotating the region under the graph

$$y = \sqrt{25 - x^2} \qquad 3 \le x \le 5$$

about the y-axis. (This is the graph of part of a circle of radius 5 centred at the origin.) Then we can double this to get the volume we want. The formula for the volume of revolution about the y-axis is

$$\int_a^b 2\pi xy\,dx = \int_3^5 2\pi x\sqrt{25 - x^2}\,dx$$

Thus the required volume is

$$\begin{aligned}
2\int_3^5 2\pi x\sqrt{25 - x^2}\,dx & \qquad \text{Put } u = 25 - x^2 \qquad du = -2x\,dx \\
& \qquad x = 3 \text{ gives } u = 25 - 9 = 16 \\
& \qquad x = 5 \text{ gives } u = 0 \\
&= 2\int_{u=16}^0 2\pi x\sqrt{u}\,\frac{du}{-2x} \\
&= 2\int_{u=16}^0 -\pi\sqrt{u}\,du \\
&= -2\pi\left[\frac{2}{3}u^{3/2}\right]_{16}^0 \\
&= -2\pi\left(0 - \frac{2}{3}(16)\sqrt{16}\right) \\
&= \frac{128\pi}{3}
\end{aligned}$$

9. **(a)** State the *linear approximation formula* for a function $y = f(x)$ near a point $x = a$ and use it to compare the true value of $f(x) = \sqrt{9 + x^2}$ for $x = 4.3$ with the value you get by using linear approximation centered at $x = 4$.

Solution: The *linear approximation formula* states that if a function $y = f(x)$ is differentiable at $x = a$, then

$$f(x) \cong f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

To use it for $f(x) = \sqrt{9 + x^2}$ centered at $x = 4$ we take $a = 4$ and use the formula. We need the derivative

$$f'(x) = \frac{1}{2\sqrt{9 + x^2}}(2x) = \frac{x}{\sqrt{9 + x^2}}$$

and the values

$$f(4) = \sqrt{9 + 16} = \sqrt{25} = 5 \qquad f'(4) = \frac{4}{5}$$

So, by linear approximation we have

$$f(x) = \sqrt{9+x^2} \cong 5 + \frac{4}{5}(x-4) \text{ for } x \text{ near } 4.$$

Putting $x = 4.3$ we find that

$$f(4.3) \cong 5 + \frac{4}{5}(4.3-4) = 5 + \frac{4}{5}(0.3) = 5 + \frac{1.2}{5} = 5.24$$

The correct value is

$$f(4.3) = \sqrt{9 + (4.3)^2} = 5.2430907 \text{ (by calculator)}$$

Comparing the result from linear approximation with the correct value, we see that there is an error of 0.0030907, or we might express this as a relative error

$$\text{relative error} = \frac{\text{approximate} - \text{true value}}{\text{true value}} = \frac{-0.0030907}{5.2430907} = 0.000589$$

so that the error is less than 1% of the true value.

(b) Explain how the linear approximation formula is used to derive the formula for *Newton's method*. Use Newton's method to find a solution of $e^x = 10x$ in the interval $[0, 1]$ to 3 decimal places.

Solution: Newton's method is an iterative method which has the purpose of finding a root to an equation $f(x) = 0$ accurately. It also aims to be efficient. We begin with such an equation $f(x) = 0$ and a 'guess' $x = x_0$ for a solution.

For the first iteration in trying to improve the guess we temporarily replace the equation $f(x) = 0$ by the linear approximation

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

centered at $x = x_0$. (For this to be possible, the derivative f' must exist at the point.) This linearised equation can be solved very easily and we get

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The key step in Newton's method is to replace our previous guess (or approximation to the solution) $x = x_0$ by the new value of x arrived at as above. Calling the next guess (or approximation) x_1, we take

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Then we repeat the process again starting with x_1 to arrive at x_2, repeat again starting at x_2 to get x_3, etc. In general we get a sequence of 'improved' approximations $x_1, x_2, x_3, \ldots$ to the solution where each one is related to the previous one via

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n = 0, 1, 2, \ldots).$$

The idea of the method is to keep calculating $x_1, x_2, x_3, \ldots$ until there is virtually no difference between x_{n+1} and x_n (or until it is clear that no such stabilisation of values is going to occur, in which case the method fails — at least for the chosen initial guess x_0). If it did happen that $x_{n+1} = x_n$ exactly, then $f(x_n) = 0$ and we have an exact solution to the equation $f(x) = 0$. In practice, we do the computations numerically (hence only approximately) and we will never be able to check that $x_{n+1} = x_n$ exactly, but it may happen that they are equal within acceptable accuracy. This does not guarantee that we have a solution with acceptable accuracy, but in many cases we will get a close approximation to a solution.

Now, to use the method on the equation $e^x = 10x$, we must first rewrite the equation in the form $f(x) = 0$. which we can do by taking $f(x) = e^x - 10x$. Then $f'(x) = e^x - 10$ and the Newton's method formula becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 10x_n}{e^{x_n} - 10} \qquad (n = 0, 1, 2, \ldots).$$

To generate an initial guess x_0, we could choose it at random, but a more systematic method is to note that $f(0) = e^0 = 1 > 0$ while $f(1) = e - 10 < 0$ and so there must be a solution in the range $0 < x < 1$. We could then choose $x_0 = 0.5$ and then we get

n	x_n	$f(x_n) = e^{x_n} - 10x_n$	$f'(x_n) = e^{x_n} - 10$	x_{n+1}
0	0.5	-3.35128	-8.35128	0.987107
1	0.987107	0.11664	-8.89625	0.111822
2	0.111822	0.0000952839	-8.88169	0.111833
3	0.111833	6.4355×10^{-11}	-8.88167	0.111833

10. **(a)** A loaded die has the following probabilities of showing the numbers 1-6 after a throw:

$$\frac{2}{16}, \frac{4}{16}, \frac{2}{16}, \frac{3}{16}, \frac{1}{16}, \frac{4}{16}$$

(in that order). Find the probability that an even number will show after the die is thrown.

Solution: We are looking for the probability of the event $\{2, 4, 6\}$ and we get that by adding up the probabilities of the individual outcomes in that event (2, 4 and 6). So we get

$$P(2) + P(4) + P(6) = \frac{4}{16} + \frac{3}{16} + \frac{4}{16} = \frac{11}{16}.$$

A random variable X associated with the experiment of rolling the die has the values

$$X(1) = 5,\ X(2) = 2,\ X(3) = 6,\ X(4) = -2,\ X(5) = 0,\ X(6) = 2.$$

Find the mean and the variance for this random variable.

Solution: The *mean* is

$$\begin{aligned} \mu &= P(1)X(1) + P(2)X(2) + \cdots + P(6)X(6) \\ &= \frac{2}{16}(5) + \frac{4}{16}(2) + \frac{2}{16}(6) + \frac{3}{16}(-2) + \frac{1}{16}(0) + \frac{4}{16}(2) \\ &= \frac{1}{16}(10 + 8 + 12 - 6 + 0 + 8) = \frac{32}{16} = 2 \end{aligned}$$

The variance is

$$\begin{aligned} \sigma^2 &= P(1)(X(1)-\mu)^2 + P(2)(X(2)-\mu)^2 + \cdots + P(6)(X(6)-\mu)^2 \\ &= \frac{2}{16}(5-2) + \frac{4}{16}(2-2) + \frac{2}{16}(6-2) + \frac{3}{16}(-2-2) + \frac{1}{16}(0-2) + \frac{4}{16}(2-2) \\ &= \frac{1}{16}(18 + 0 + 32 + 48 + 4 + 0) = \frac{102}{16} = \frac{51}{8} \end{aligned}$$

(b) A factory produces bottles of sauce that are sold as 2.5 litre bottles. A good model is that the quantity of sauce in a bottle obeys a normal distribution with mean 2.55 (litres) and standard deviation 0.17. What proportion of the bottles have less than 2.5 litres in them?

Solution: We are told that the bottles obey a normal distribution with mean $\mu = 2.55$ and $\sigma = 0.17$. Thus the probability that there will be less that 2.5 litres in a bottle is

$$\begin{aligned} \Phi_{\mu,\sigma}(2.5) &= \Phi_{0,1}\left(\frac{2.5 - 2.55}{0.17}\right) \\ &= \Phi_{0,1}\left(\frac{-0.05}{0.17}\right) \\ &= \Phi_{0,1}(-0.2841) \end{aligned}$$

(in terms of a standard normal $\Phi_{0,1}$). As the tables give values of $\Phi_{0,1}(z)$ for $z > 0$, we must exploit the symmetry of the standard normal to get

$$\Phi_{0,1}(-0.2841) = 1 - \Phi_{0,1}(0.2841) = 1 - 0.6156 = .3843$$

Thus 0.3843 (or 38.43%) of the bottles will have less that 2.5 litres in them.