Chapter 7: Applications of Integration

Course 1S3, 2006–07

May 11, 2007

These are just summaries of the lecture notes, and few details are included. Most of what we include here is to be found in more detail in Anton.

7.1 Remark. The aim here is to illustrate that integrals (definite integrals) have applications to practical things. Quite a few concepts in scientific theories are explained in terms of integrals, but to go into them would require a knowledge of the necessary background.

Some of the material that was covered in this chapter in previous years (areas, arc length, volumes of revolution) were covered in 1S1 this year. They are relegated to the appendix.

We will consider a number of applications — fluid pressure, work, and centre of mass.

7.2 Fluid Pressure. We now explain an application of integration to fluid pressure. First we need some idea of what a fluid is and what we mean by pressure. Then a little about how to work with it.

Roughly a fluid is a liquid or a gas. They have the property that they adapt their shape to the shape of any container they are in. The fluid will then exert a force on the wall of the container. In the case of a gas (like the air in a tyre) the pressure comes from the internal energy of the gas (the molecules of gas will be moving around and causing the pressure by bouncing off the walls). In the case of a fluid, we are thinking of gravity making the fluid settle to the bottom of the container and we are not considering increased pressure that might come from a pump or other mechanism.

We will deal with the fluid case. To avoid complications we will assume we have a uniform incompressible fluid (say water or oil, but not a mixture where there could be layers of different types). The *pressure* exerted by the liquid is *force per unit area*.

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

To get the pressure at a point (say a point on the wall of the container) we should measure the force on a tiny section of the wall and divide by the area of that tiny section. In this way we get the pressure at that point and not the average over a bigger area.

The key properties are that the pressure depends only on the depth (distance down from the free surface of the liquid) and does not depend on direction. Even if the link to the surface is not direct (for example a water tap linked to a tank upstairs by a pipe that goes under the floor), the pressure still depends only on the depth and not on the volume of liquid above the point.

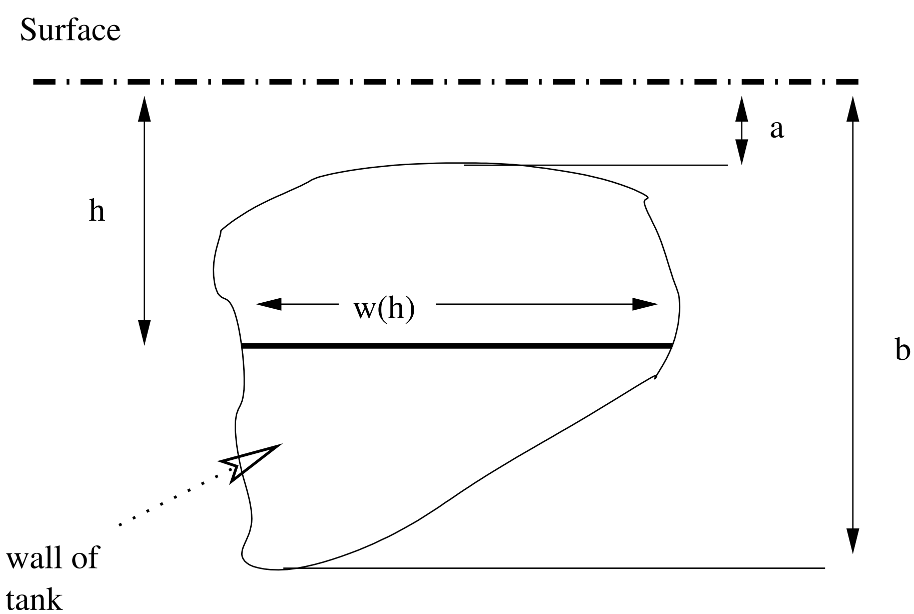
We can work this out in terms of the strength of gravity (the acceleration due to gravity g if you like) and the density of the liquid. Density means mass per unit volume and we could perhaps denote the value of this for our liquid by δ. However it is the gravitational force that arises from the liquid that really counts for the force and this will be δg in units of Force/volume (weight force per unit volume). We call this quantity ρ and refer to it as the *weight density* of the liquid. The assumption that this is constant throughout the liquid translated into the fact that the pressure at depth h down from the free surface of the liquid is

$$\text{Pressure} = P = \rho h$$

This is the force per unit area caused by the pressure of the liquid acting on a wall of the container at a given depth h.

Now we consider a whole section of the wall of the container and try to figure out the total force on that (caused by the liquid). We cannot just multiply the pressure P by the area of the wall because the pressure varies with depth.

We think of a scenario as pictured here, where we have a vertical wall (that makes things simpler). The wall may be far below the surface of the liquid, or it may start at the surface. (Any bits of the wall that are not submerged are not going to matter as there will be no pressure on those bits.)



To figure out the total pressure, we look at a specific depths h, or to avoid having zero area, we consider the part of the wall between depths h and $h + dh$. Then we work out the force on that strip (constant depth h and so constant pressure $P = \rho h$ at that depth) to get

$$\text{Force} = P \times \text{area} = \rho h \times \text{area}$$

To find the area of that horizontal strip, we treat it as a rectangle of total width $w = w(h)$ and depth dh. So

$$(\text{area between depths } h \text{ and } h + dh) = w(h)\,dh$$

Notice that this width $w(h)$ could vary as the depth changes, depending on the shape of the wall. We get

$$(\text{Force on strip between depths } h \text{ and } h+dh) = \rho h w(h)\, dh$$

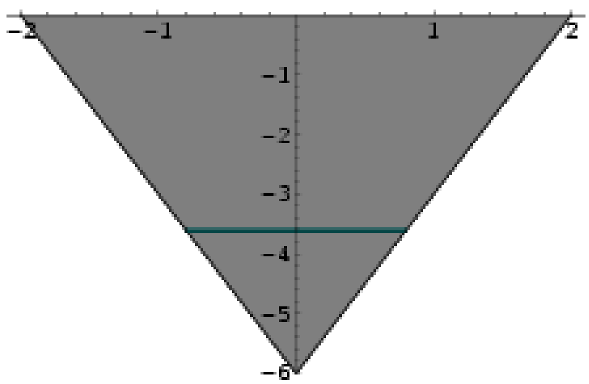
and we add these up to get

$$\text{Total pressure force} = \int_{h=a}^{b} \rho h w(h)\, dh$$

To review all the notation, $h =$ the depth under the surface of the liquid. We are assuming the liquid has uniform weight density ρ, the wall is vertical, the top of the wall is at depth $h = a$, the bottom at depth $h = b$ and the width of the wall at depth h is $w(h)$.

7.3 Example. The (vertical) end wall of a trough is in the form of an isosceles triangle with one the point down. The top of the wall measures 4 units wide and the depth is 6 units. Find the pressure force on the wall caused by the liquid in the trough when the trough is just full to the brim. (Your answer will be in terms of ρ the weight density of the liquid, which we assume to be uniform.)

Solution: Here is a picture of the situation



with a cross section drawn at depth h (between h and $h + dh$. We need a formula for the width $w(h)$ at depth h and we can get this by using similar triangles. The ratio of the width to depth of the whole triangle is $4/6$ and that will be the same for the lower triangle. The lower one has depth $6 - h$ and so

$$\frac{4}{6} = \frac{w(h)}{6-h} \Rightarrow w(h) = \frac{2}{3}(6-h)$$

The formula gives

$$\text{Pressure force } = \int_0^6 \rho h w(h)\, dh = \int_0^6 \frac{2\rho}{3} h(6-h)\, dh = \int_0^6 \frac{2\rho}{3}(6h - h^2)\, dh$$

and this works out $\quad\ldots\quad$ as 24ρ.

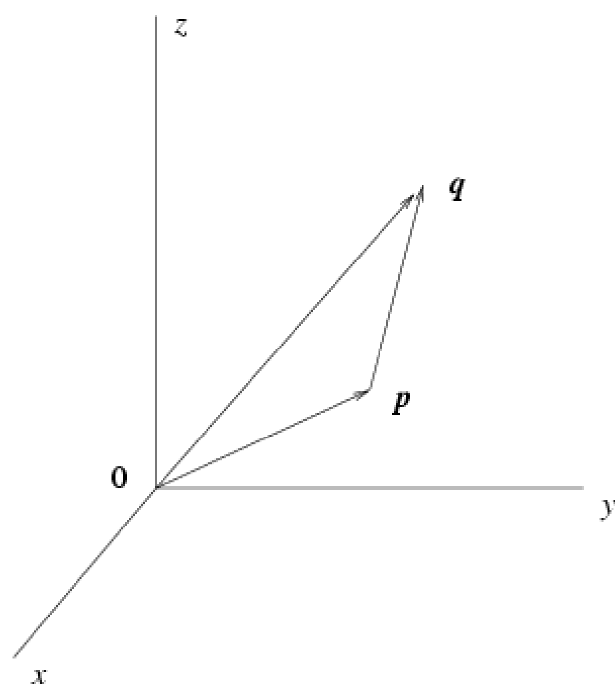
7.4 Work. Work is a technical term, with a precise meaning.

First, consider a constant force acting on an object (we think of a particle or object concentrated at a point to avoid complications). Say the particle moves from one position to another. Then there is a dot product formula (using vectors) for the work done **by** the force **on** the object.

Let's explain this from the beginning. We're thinking in space (3 dimensions) for now, as this is realistic. (Later we will make calculations in only one dimension as this is really all we know enough to cope with for now.) A force is represented by a vector (think of an arrow in space) because it has a direction and a magnitude (or strength). Say the force is given by $\mathbf{F}$ (or $\vec{F}$ if you prefer a different notation for a vector). We won't worry about the units, but we should assume we are using a consistent set of units.

We can represent positions of points in space by coordinates (x, y, z) with respect to 3 (preselected and fixed) perpendicular axes meeting at a chosen origin. Or we can use the so-called position vector with components x, y and z in the directions of the 3 axes. So the position vector of a point $\mathbf{p} = (x, y, z)$ is the vector $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $\mathbf{i}$ is the unit vector in the direction of the x-axis, $\mathbf{j}$ is the unit vector in the direction of the y-axis, and $\mathbf{k}$ is the unit vector in the direction of the z-axis. Once we get used to the idea that the same data is needed to specify a point as is needed to specify a vector (3 numbers x, y and z which we may regard as coordinates of the point or components of the vector) we realise quite quickly that it is convenient to switch points for their position vectors when it suits us.

Now if a particle moves form position $\mathbf{p}$ to position $\mathbf{q}$, then its *displacement* is the vector represented by the arrow from where it started $\mathbf{p}$ to where it ended $\mathbf{q}$. In vector algebra this displacement is the vector $\mathbf{q} - \mathbf{p}$, as you can work out from a picture like this:



This sort of manipulation of vectors should be familiar from 1S2.

The definition of the work done by a constant force $\mathbf{F}$ on a particle is given by the dot product (sometimes called the scalar product)

$$\text{Work} = \text{Force} \cdot \text{Displacement}$$

or

$$\text{Work} = \mathbf{F} \cdot (\mathbf{q} - \mathbf{p})$$

if $\mathbf{p}$ is the starting position and $\mathbf{q}$ the ending position.

Maybe it is no harm to recall a few little things about dot products. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors we can work out their dot product easily from

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

(multiply the matching components and add up the answers so that the result is a scalar). You can also describe the number in a geometrical way, if you think in terms of the angle θ between the two vectors:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

(where $\|\mathbf{a}\|$ means the length or magnitude of the vector $\mathbf{a}$, which is given in components by $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, and of course $\|\mathbf{b}\|$ is the length of $\mathbf{b}$).

Maybe a better notation is to use $\Delta\mathbf{p}$ for the change in position, and then we get the formula

$$\boxed{\text{Work} = \text{Force} \cdot \text{Displacement} = \mathbf{F} \cdot \Delta\mathbf{p}}$$

That is for a constant force. Suppose now that the force is changing as the particle moves. So the force is perhaps a function $\mathbf{F}(t)$ of time t (measured say in seconds from some starting time) and the particle has position $\mathbf{p}(t)$ at time t. In order to be able to plausibly use the formula for a constant force, what we do is think of dividing the elapsed time into really tiny time intervals from t to $d + dt$. If dt is really small (think of it as infinitesimal) then the force should be more or less constant $\mathbf{F}(t)$ during that short time and the work done by the force in that short time should be

$$\text{Work} = \text{Force} \cdot \text{Displacement} = \mathbf{F}(t) \cdot (\mathbf{p}(t + dt) - \mathbf{p}(t))$$

We will write

$$d\mathbf{p} = \mathbf{p}(t + dt) - \mathbf{p}(t)$$

and the work done over the short interval as

$$\text{Work} = \mathbf{F}(t) \cdot d\mathbf{p}$$

We then have to add up all the little bits of work to get the total work done. If the starting time is $t = t_0$ and the ending time is $t = t_1$ we get an integral (rather than a sum)

$$\boxed{\text{Work} = \int_{t=t_0}^{t=t_1} \mathbf{F}(t) \cdot d\mathbf{p}}$$

This is an official definition, more or less 100% accurately stated, but it brings up a kind of integral that we will not deal with until next years course. So we will only look at a rather special case, where the motion takes place along a single axis (the x-axis) and the force is also one dimensional. That means that the force vector can only point right (in the positive direction) or left (negative direction) and is given by a number with a plus or minus sign. So we don't really have vectors any more, as the position is given by a single coordinate x.

We will further simplify things and suppose that the force depends only on the position x, so that the force is $F(x)$. Now the formula can be simplified because we can do the integral in the x variable (by substitution, or change of variable, from time t). The formula becomes

$$\text{Work} = \int_{x=x_0}^{x=x_1} \mathbf{F}(x) \cdot dx$$

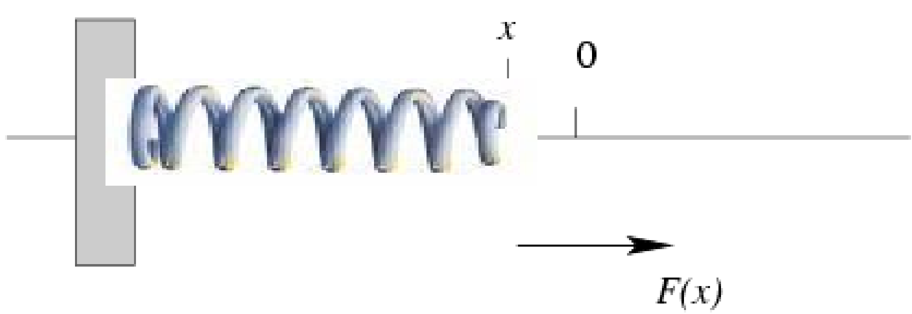
for the work *done by the force $F(x)$ on the particle when the particle moves from starting place $x = x_0$ to ending place $x = x_1$*.

Hooks Law. There is a realistic situation where a force $F(x)$ of this kind does arise in practice. Suppose the force is provided by a spring attached at one end to the particle and anchored at the opposite end. We suppose the whole business is in one dimension so that the particle can only move back and forth along the x-axis and the spring is anchored somewhere on the axis. Hooks law says that the force exerted by a spring is proportional to the amount the spring is compressed or expanded from its natural rest length. (This law is good provided you don't stretch or squeeze the spring too far — if you do that it would eventually get distorted and not go back to its original shape.)

To make life easy, say the spring is anchored somewhere to the left of the origin and the rest length of the print leaves the free end at the origin $x = 0$. Then the position x of the end tells you just how much the spring is expanded (if $x > 0$) or compressed (if $x < 0$). Hooks Law says then there is a constant κ (called the spring constant) so that the force exerted by the spring when the end is at x is

$$F(x) = -\kappa x$$

Here $\kappa > 0$ and you should be able to work out that the spring will be pulling the particle back to the left when $x > 0$. That is the reason for the minus sign, so that $F(x) < 0$ when $x > 0$ (and $F(x) > 0$ when $x < 0$ because the spring will be pushing to the right when $x < 0$, meaning that the spring is compressed).



For this example, if we suppose the particle moved from the position where $x = 0.5$ to the position where $x = 1$, the work done by the spring on the particle would be

$$\begin{aligned} \text{Work} &= \int_{0.5}^{1} F(x)\,dx \\ &= \int_{0.5}^{1} -\kappa x\,dx \\ &= \left[-\frac{\kappa}{2} x^2 \right]_{0.5}^{1} \\ &= -\frac{\kappa}{2} - \left(-\frac{\kappa}{2}(0.5)^2 \right) \\ &= -\frac{3}{8}\kappa \end{aligned}$$

Note first that we would need to know the spring constant κ to get the actual value here. Second, note that it is perfectly possible for an amount of work to be negative (as it is here). It makes sense because the spring would be pulling in the opposite direction to the motion (as the spring is extended when $x > 0$) and so the particle is moving against the force.

You might ask how this could happen? Well, there are at least tow possible reasons. One is that there is another force we have not mentioned (doing a positive amount of work against the spring). Another is that the particle could have an initial momentum, and, even though slowed down by the spring, could still move against the spring. In fact, you could be looking at an oscillating thing, where the particle is oscillating back and forth past the equilibrium position at $x = 0$. If we just look at a part of the motion, we could be in the situation above.

It would also make sense to as for the work done if the particle moved from $x = 1$ to $x = 0.5$. This would be

$$\begin{aligned} \text{Work} &= \int_{1}^{0.5} F(x)\,dx \\ &= \int_{1}^{0.5} -\kappa x\,dx \\ &= \left[-\frac{\kappa}{2} x^2 \right]_{1}^{0.5} \\ &= -\frac{\kappa}{2}(0.5)^2 - \left(-\frac{\kappa}{2} \right) \\ &= \frac{3}{8}\kappa \end{aligned}$$

7.5 Centres of mass. For some purposes the position of a solid object is described adequately by a single point, the centre of mass, and the object can be treated as a point mass at that point. For example, if considering gravity caused by the sun, we may treat the sun as a point mass concentrated at its centre. The gravitational force depends on the mass of the sun and the distance to the centre of mass. The same is true of the moon or the earth, but this is not adequate if we want to think about the rotation of any of these around their centres of mass.

If we look at a more mundane solid object like a golf ball, a pencil, or a rugby ball, the centre of mass is characterised by this: if you support the object on a knife edge where the edge passes directly under the centre of mass, the object should in principle balance. For example, a pencil is probably symmetrical around its central axis and should (theoretically) balance if you put it down on its point with the pencil exactly vertical. In practice, this would be tricky because it would be so unstable even if you could get it exactly vertical.

Working out the position of the centre of mass of a solid object requires triple integrals, and we have not learned about them. So we will concentrate on some rather simple examples that can be treated in one dimension. Think of a straight wire or rod, which is very thin (so that we can think of it as 1 dimensional) but is maybe of varying thickness. If it is not of varying thickness, the centre of mass would be at the middle point, but if there is some variation, the centre of mass could be somewhere else. Or think of a pencil (a nice round one that is even pared so that it is symmetrical around its axis). The centre of mass would not be exactly half way along because you need to take account of the fact that it is thinner at the pointed end. So you should be able to balance it somewhere a little closer to the blunt end than to the pointed end.

An example that is maybe quite realistic is a billiard cue.

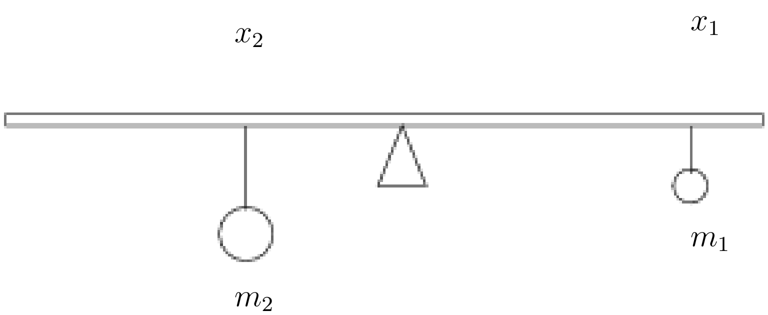
It is fairly clearly symmetrical around its axis and the center of mass should be somewhere along the axis. We want to work out how far from one of the ends it is.

The condition for something to balance is described via moments. If you have a balanced support rod, supported at one point and you hang a mass m off it at distance x from the support point, the propensity of the thing to tip over is measured my the moment

$$\text{moment} = mx$$

If you have two masses, m_1 supported at position x_1 and m_2 supported at position x_2 (away from the support point) the total turning effect is measured by the total moment

$$\text{moment} = m_1x_1 + m_2x_2.$$



Notice that, by using a coordinate x for the position, which can be positive to the right and negative to the left, we get a total clockwise moment from this formula. The condition for the arrangement to balance is that the moment should be 0.

Next, we can do a similar thing with 3 or more suspended masses. If we have n of them at positions $x_1, x_2, \ldots, x_n$ with corresponding masses $m_1, m_2, \ldots, m_n$ we get

$$\text{moment} = m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

To apply this to a solid object, we imagine that the solid thing is composed of little tiny masses of mass dm at distance between x and $x + dx$ away from the support point. We no longer really need a balance beam to hang these things from as the solid should keep itself together (if it is really solid). The moment caused by the little tiny mass dm at distance x is

$$x\,dm$$

and we should add these up to get a total moment. This becomes some sort of integral

$$\text{moment} = \int x\,dm$$

extending over the length of the object. We are used to integrals with dx at the end and to get that we use the idea of linear density.

The *linear density* of a 1-dimensional object (like a straight wire of varying thickness, or think of the billiard cue ignoring its thickness) is its **mass per unit length**. If we take the total mass divided by the total length we get that the average linear density is

$$\text{average linear density} = \frac{\text{total mass}}{\text{total length}}.$$

But to get an idea of the linear density at a particular place we should do this calculation over a very short length. The shorter the better and in the limit we have an almost infinitesimally short length from x to $x + dx$ with mass dm. We get the linear density function

$$\sigma(x) = \frac{dm}{dx}$$

Thus

$$dm = \sigma(x)\,dx$$

and the total (clockwise) moment is

$$\text{moment} = \int x\,dm = \int x\sigma(x)\,dx.$$

We should integrate this between appropriate limits. That is, if the object stretches from $x = a$ to $x = b$, we have

$$\text{moment} = \int_{x=a}^{x=b} x\sigma(x)\,dx.$$

The condition for the object to balance when supported at $x = 0$ is that the moment is 0.

Suppose we work out the integral and we don't get 0. Then we know the centre of mass is not at position $x = 0$. However, all is not lost because if the centre of mass is at $x = \bar{x}$, then the moment should be 0 if we support the object at $x = \bar{x}$. The condition for this works out as

$$\int_{x=a}^{x=b} (x - \bar{x})\,dm = \int_{x=a}^{x=b} (x - \bar{x})\sigma(x)\,dx = 0.$$

We can rearrange this as

$$\begin{aligned}\int_{x=a}^{x=b} x\,dm - \bar{x}\int_{x=a}^{x=b} dm &= 0\\ \bar{x}\int_{x=a}^{x=b} dm &= \int_{x=a}^{x=b} x\,dm\\ \bar{x} &= \frac{\int_{x=a}^{x=b} x\,dm}{\int_{x=a}^{x=b} dm}\end{aligned}$$

The integral $\int_{x=a}^{x=b} dm$ represents the total mass of the object (add up the masses of all the little sections) and so we get

$$\bar{x} = \frac{\text{total moment}}{\text{total mass}}.$$

If we express this in terms of the linear density function $\sigma(x)$ we get

$$\bar{x} = \frac{\int_{x=a}^{x=b} x\sigma(x)\,dx}{\int_{x=a}^{x=b} \sigma(x)\,dx}$$

as our final formula for the position of the centre of mass.

7.6 Example. Suppose a billiard cue is made of the same material throughout (constant mass per unit volume, or density) and it is tapered so that the radius is

$$r(x) = 0.02 - \frac{x}{100} \qquad (0 \leq x \leq 1).$$

(We could say it is 1 metre long, radius 2cm at the handle, radius 1cm at the tip and it tapers linearly. That might be a less mathematical way to express the same formula.) Find its centre of mass.

We need the linear density $\sigma(x)$ and we can work that out fairly easily in terms of the actual density. We are told the actual density is constant, call it ρ. Since we are not told ρ it seems we have to keep this unknown quantity around (but it will actually cancel out at the end).

To get the linear density at x, think about a slice through the billiard cue from x to $x + dx$. It is almost a cylinder of radius $r(x)$, 'height' (or length) dx. If we use the formula $\pi r^2 h$ for the volume of a cylinder we get that the little piece has

$$\text{volume} = dV = \pi r(x)^2 dx$$

mass

$$\text{mass} = \text{density} \times \text{volume} = \rho dV = \pi\rho r(x)^2 dx$$

and so the linear density is

$$\sigma(x) = \pi\rho r(x)^2.$$

We want to calculate

$$\begin{aligned}\bar{x} &= \frac{\int_{x=0}^{x=1} x\sigma(x)\,dx}{\int_{x=0}^{x=1} \sigma(x)\,dx} \\ &= \frac{\int_{x=0}^{x=1} \pi\rho x r(x)^2\,dx}{\int_{x=0}^{x=1} \pi\rho r(x)^2\,dx}\end{aligned}$$

These integrals are not very complicated

$$\begin{aligned}\int_{x=0}^{x=1} \pi\rho x r(x)^2\,dx &= \pi\rho \int_0^1 x(0.2 - 0.1x)^2\,dx \\ &= \pi\rho \int_0^1 x(0.04 - 0.04x + 0.01x^2)\,dx \\ &= \pi\rho \int_0^1 0.04x - 0.04x^2 + 0.01x^3\,dx \\ &= \pi\rho \left[0.04\frac{x^2}{2} - 0.04\frac{x^3}{3} + 0.01\frac{x^4}{4}\right]_0^1 \\ &= \pi\rho(0.02 - 0.013333 + 0.0025) - 0 = \pi\rho(0.009167) \\ \int_{x=0}^{x=1} \pi\rho r(x)^2\,dx &= \pi\rho \int_0^1 (0.04 - 0.04x + 0.01x^2)\,dx \\ &= \pi\rho \left[0.04x - 0.04\frac{x^2}{2} + 0.01\frac{x^3}{3}\right]_0^1 \\ &= \pi\rho(0.04 - 0.02 + 0.003333) - 0 = \pi\rho(0.02333)\end{aligned}$$

This leads to

$$\bar{x} = \frac{\pi\rho(0.009167)}{\pi\rho(0.02333)} = \frac{0.009167}{0.02333} = 0.392877$$

(Note that the fact that it is less than $1/2$ is plausible.) If we did the calculation using

$$r(x) = (2 - x)/100$$

we would get the answer $\bar{x} = 11/28$.

Appendix

7A.7 Areas. Since we explained the idea of a definite integral using areas as a guide, we are not going too far here. Reiterating what we had in the previous chapter:

> If $f(x) \geq 0$ for $a \leq x \leq b$, then the area of the region in the plane bounded by the graph $y = f(x)$, the x-axis $y = 0$ and the two vertical lines $x = a$ and $x = b$ is
>
> $$\int_a^b f(x)\,dx$$

We called this the 'area under the graph' before. Just note that the integral is not an area if $f(x)$ is ever negative.

7A.8 Areas (between two graphs)**.** Suppose now $f(x) \geq g(x)$ for $a \leq x \leq b$. Consider the region in the plane bounded on top by the upper graph $y = f(x)$, below by the lower graph $y = g(x)$ and on each side by the the two vertical lines $x = a$ and $x = b$.

More formally it is

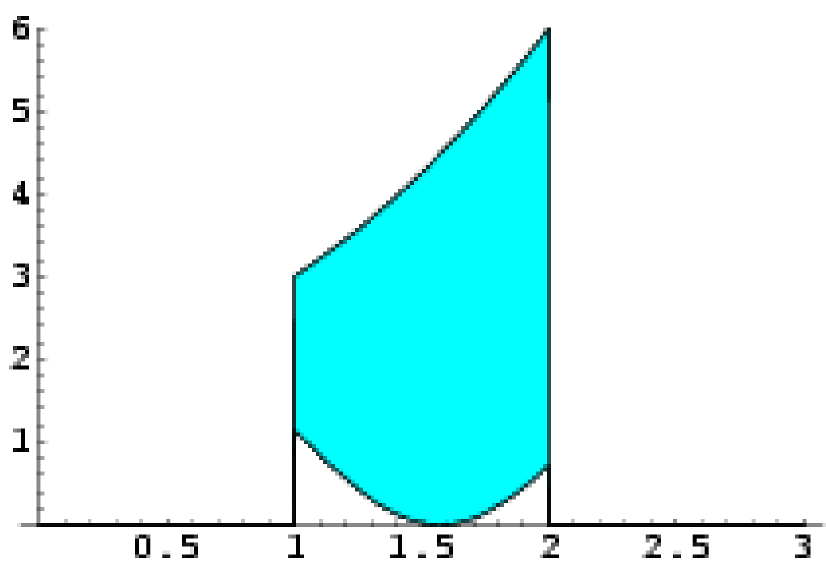
$$\{(x, y) : a \leq x \leq b, g(x) \leq y \leq f(x)\}$$

The formula for the area of this region is

$$\int_a^b f(x) - g(x)\,dx = \int_a^b (\text{bigger} - \text{smaller})\,dx$$

This is easy to see when the lower graph is always positive ($g(x) \geq 0$ for $a \leq x \leq b$) because we then can just use subtraction

$$\text{area between} = (\text{area below } y = f(x)) - (\text{area below } y = g(x)) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$



But the argument gets a small bit trickier if one or both of the functions is sometimes negative.

There is a pseudo-explanation for this formula using the notion of *infinitesimals*. We use it because it is quite a convenient way to explain the formulae we will have later.

The idea behind infinitesimals is a little suspicious. (There is a very fancy way to make it all ok, but we will just do it in the way that looks doubtful but seems to work. Without using the fancy theory to make it ok, there is another way by taking limits of Riemann sums.) The idea with infinitesimals is to try and short circuit the necessity foe using limits to explain

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

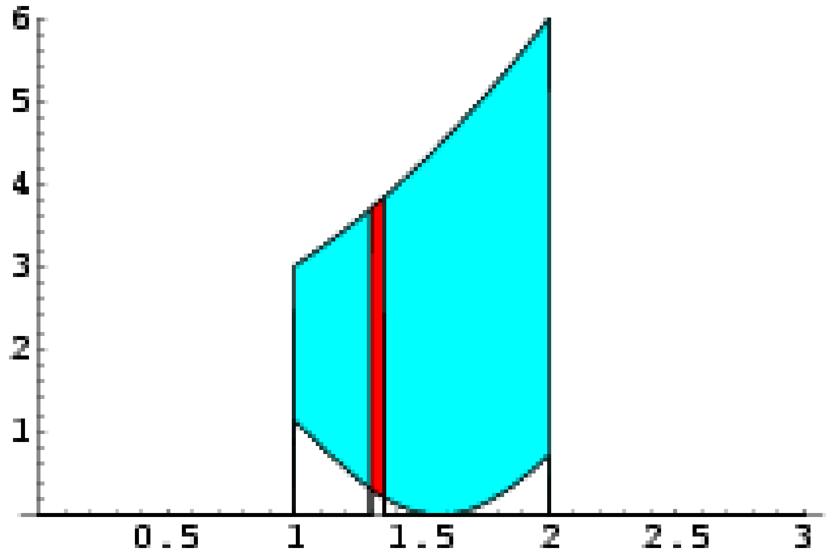
and in

$$\int_a^b f(x)\,dx = \lim \left(\text{Riemann sums } \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \right) = \lim \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

by imagining we could replace a very small Δx with an unbelievably small dx. Now we can't take $dx = 0$ or we would get nothing good, but the idea of an infinitesimal is to take a "number" dx which is not zero and is yet so small that it is smaller than anything you can think of.

Now, in ordinary numbers, there is no such infinitesimal quantity. The notations $\frac{dy}{dx}$ and $\int_a^b f(x)\,dx$ are essentially based on the intuition that there is such a thing (at least in some sort of limit). Just as the Chain rule and substitution work out nicely when you assume that dy, dx and du can be manipulated like numerical quantities, so we can explain integration as a sort of 'sum' of $f(x)\,dx$. The only thing is that when $\Delta x_i = (x_i - x_{i-1})$ are small quantities (even very small) we have a finite number of steps to get from $x = a$ to $x = b$. When we use infinitesimal steps, we have to imagine being able to take infinitely many. The notation $\int$ for integrals comes from S for sum and it is modified to reflect the fact that we are dealing with a kind of 'sum' out of the ordinary.

Going back to the example of explaining the formula for the area between the two curves, we think about dividing the area into many infinitesimally thin vertical strips. Look at one such strip, between x and $x + dx$.



Since it is so thin, we can look at it as a rectangle — with width $= dx$ and height = the difference in heights between the upper graph $y = f(x)$ and the lower one $y = g(x)$. This difference in heights is $f(x) - g(x)$ (and that is correct even when one or both heights is negative). So

$$\text{area of strip } = (f(x) - g(x))\, dx$$

'Add' these up to get the total area between the two curves as

$$\int_{x=a}^{b} (f(x) - g(x))\, dx.$$

You need to go from $x = a$ to $x = b$ to count all the area.

7A.9 Examples. (i) Find the area between the graphs $y = x^2 + 2$ and $y = 1 - x^4$ for $-1 \leq x \leq 2$.

Solution:

$$\int_{-1}^{2} (x^2 + 2) - (1 - x^4)\, dx$$

and this works out as $\ldots$ $63/5$.

(ii) Find the area of the bounded region of the plane bounded by the curves $y = x^4 - 4$ and $y = 5x^2 + 10$

Solution: We need to understand that the curves intersect. To find the points of intersection, solve

$$\begin{aligned} x^4 - 4 &= 5x^2 + 10 \\ x^4 - 5x^2 - 14 &= 0 \\ (x^2 - 7)(x^2 + 2) &= 0 \end{aligned}$$

So $x^2 - 7 = 0$ *or* $x^2 + 2 = 0$. But $x^2 = -2$ is not possible (for real numbers x) and so we are left with $x^2 = 7$, thus $x = \pm\sqrt{7}$.

So the two curves cross just twice and between $x = -\sqrt{7}$ and $x = \sqrt{7}$. One will be above the other in between (and that is where the bounded region comes from). To see which one is above, take a convenient value of x in the range $-\sqrt{7} < x < \sqrt{7}$ such as $x = 0$ and we see that $y = 5x^2 + 10$ is the upper one ($y = 10$ at $x = 0$) while $y = x^4 - 4$ is the lower one.

So our formula for the area gives

$$\int_{-\sqrt{7}}^{\sqrt{7}} (\text{bigger} - \text{smaller})\, dx = \int_{-\sqrt{7}}^{\sqrt{7}} (5x^2 + 10) - (x^4 - 4)\, dx$$

and this works out as $\quad \ldots \quad 476\sqrt{7}/15$

7A.10 Length of a curve (case of a graph $y = f(x)$)**.** We now investigate the length of a graph

$$y = f(x) \qquad (a \le x \le b).$$

What we mean now is the *curved length*, meaning the distance you would travel if you walked along a road shaped like the graph, or the length of a piece of thread that could be used to make the shape of the graph.

We **do not mean** the straight line distance from the beginning point $(x, y) = (a, f(a))$ to the end point $(x, y) = (b, f(b))$. We do mean the length along the curve.

Our approach to finding the formula is to again divide the curve up into infinitesimal segments. The one from $(x, y) = (x, f(x))$ to $(x, y) = (x + dx, f(x + dx))$ for example. Write $y = f(x)$ and $y + dy = f(x + dx)$ (or $dy = f(x + dx) - f(x)$). For the short segment, the curvature of the curve has no chance to make a difference and we do use the straight line distance from the beginning to the end to find the length of the little section. This means using the distance formula for the distance between 2 points in the plane and we get

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x + dx - x)^2 + (y + dy - y)^2} = \sqrt{(dx)^2 + (dy)^2}$$

We should 'add' up the lengths of these infinitesimal section to get the total length and we should end up with some sort of integral

$$\int_{x=a}^{x=b} \sqrt{(dx)^2 + (dy)^2}$$

However, this does not look nice as we expect integrals to have a dx at the end. To fix this we divide inside the $\sqrt{\ }$ by $(dx)^2$ and compensate by multiplying by dx outside. So we get

$$\int_{x=a}^{x=b} \sqrt{\frac{(dx)^2 + (dy)^2}{(dx)^2}}\, dx = \int_{x=a}^{x=b} \sqrt{\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now we have a formula that makes perfect sense with $dy/dx = f'(x)$ in it. Our formula is

$$\text{Length of graph} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

7A.11 Example. (i) Find the length of the graph

$$y = 1 + 4x^{3/2} \qquad (1 \le x \le 4)$$

Solution: We just need to apply the above formula. For that we need to know

$$\frac{dy}{dx} = 4\frac{3}{2}x^{1/2} = 6\sqrt{x}$$

and then we have

$$\text{Length of graph} = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 \sqrt{1 + 36x}\,dx$$

and this can be worked out easily enough ... and the answer should come out as $(1/54)(145\sqrt{145} - 37\sqrt{37})$

(ii) Find the length of the graph

$$y = \cos x \qquad \left(0 \le x \le \frac{\pi}{2}\right)$$

Solution: This seems easy enough. We need

$$\frac{dy}{dx} = -\sin x$$

and then the length is given by our formula

$$\text{Length} = \int_0^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/2} \sqrt{1 + \sin^2 x}\,dx$$

However, this integral is a bit of a challenge. *Mathematica* might be able to help:

```
In[2]:= Integrate[Sqrt[1 + Sin[x]^2], {x, 0, Pi/2}]

Out[2]= EllipticE[-1]
```

but this `EllipticE` thing looks complicated. We can ask *Mathematica* what it is:

```
In[3]:= ?EllipticE
EllipticE[m] gives the complete elliptic integral E(m). EllipticE[phi, m]
   gives the elliptic integral of the second kind E(phi|m).
```

and we have got into something too complicated for us.

There is a numerical answer to this problem

```
In[4]:= NIntegrate[Sqrt[1 + Sin[x]^2], {x, 0, Pi/2}]

Out[4]= 1.9101
```

(and that is simple enough to understand), but the problem is that there is no simple way to write the antiderivative for $\sqrt{1+\sin^2 x}$. Asking *Mathematica* for the antiderivative (without any limits in the integral), we get

```
In[5]:= Integrate[Sqrt[1 + Sin[x]^2], x]

Out[5]= EllipticE[x, -1]
```

and we will not learn what that is in this course.

Though it might seem that we just chose an unfortunate example, there are in fact almost no examples (except the one we did first and a few others) where the formula for the length of a graph results in an integral we can do by hand.

7A.12 Length of a curve (case of a parametric curve)**.** We know that not all curves are graphs. For example, a graph $y = f(x)$ of a function has at most one point on each vertical line, but there are plenty of curves like circles or ellipses or spirals that double back on themselves.

We can describe these using *parametric equations*. Think of the point of your pen moving along as it draws the curve and then record the position $(x, y) = (x(t), y(t))$ of the pen at each instant of time t. We end up with parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \qquad a \le t \le b \end{cases}$$

(where $t = a$ can be thought of as the start time and $t = b$ the end time for the drawing). You can also think of the point of the pen as simply a moving point in the plane. The problem we are considering now is how far the point has travelled as it traces out the curve. (If the point retraces some parts of the curve more than once, we will count it more than once. We will get the total distance travelled.)

Again we divide the time into short segments, from t to $t + dt$. In that (infinitesimal) time the point has moved from $(x, y) = (x(t), y(t))$ to $(x, y) = (x(t + dt), y(t + dt)) = (x + dx, y + dy)$ and we end up in a similar way as before at

$$\text{Length} = \int_{t=a}^{t=b} \sqrt{(dx)^2 + (dy)^2}$$

To make this look right we divide inside the $\sqrt{}$ by $(dt)^2$ and compensate by multiplying outside by dt. We get

$$\int_{t=a}^{t=b} \sqrt{\frac{(dx)^2 + (dy)^2}{(dt)^2}}\, dt = \int_{t=a}^{t=b} \sqrt{\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2}}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

So our ultimate formula is

$$\text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Those who know a little mechanics will recognise

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}$$

as the magnitude of the *velocity vector*

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dx}\mathbf{j}$$

and the magnitude of the velocity is called the *speed*. So the total distance travelled is the integral of the speed with respect to time.

7A.13 Example. Find the length of the parametric curve

$$\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \qquad 0 \le t \le \pi \end{cases}$$

Solution: We just have to use the formula and for that we will need

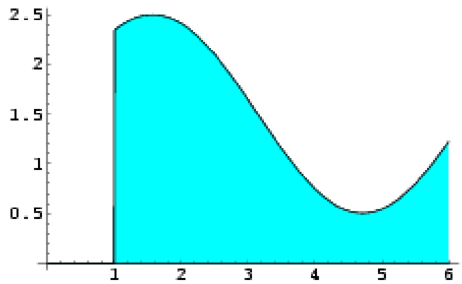
$$\begin{aligned}
\frac{dx}{dt} &= -\sin t + \sin t + t\cos t \\
&= t\cos t \\
\frac{dy}{dt} &= \cos t - \cos t + t\sin t \\
&= t\sin t \\
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{t^2\cos^2 t + t^2\sin^2 t} \\
&= \sqrt{t^2(\cos^2 t + \sin^2 t)} \\
&= \sqrt{t^2} = t \qquad (t \ge 0)
\end{aligned}$$

So our formula gives

$$\text{Length} = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dt = \int_0^\pi t\, dt = \left[\frac{t^2}{2}\right]_0^\pi = \frac{\pi^2}{2}$$

7A.14 Volumes of revolution (by slicing). Although most volumes require triple integrals to work them out (which is beyond our course this year), we can deal with some especially symmetrical shapes using single integrals. The objects (3 dimensional or solid objects) we will consider will be symmetrical about some axis but we won't consider every possible axially symmetric shape.

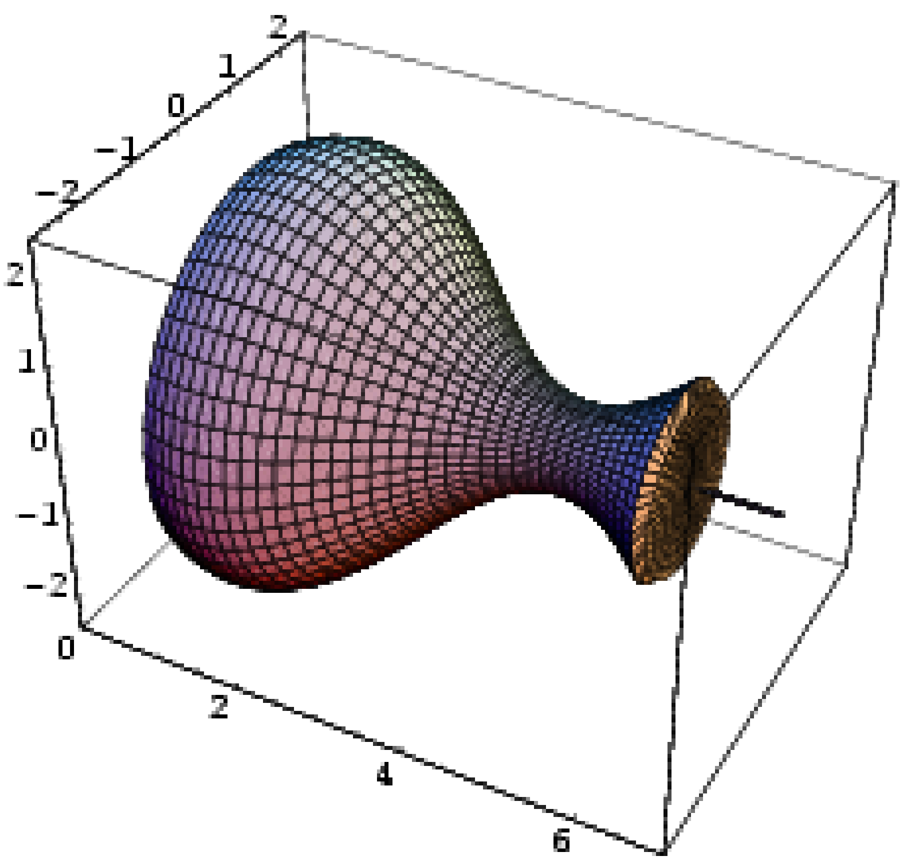
We start with the area in the plane 'under' a graph $y = f(x)$, $a \le x \le b$.



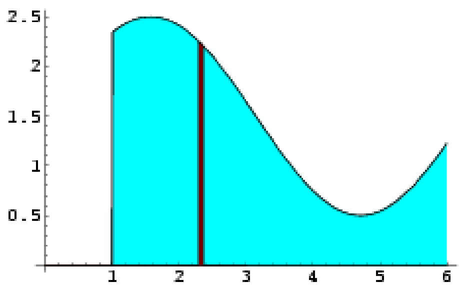
Imagine this constructed as a flat piece of cardboard and then imagine rotating it about the x-axis. As it rotates around and around it will sweep through a part of space. Our first volume formula (for this case of *rotating the area under the graph about the x-axis*) is

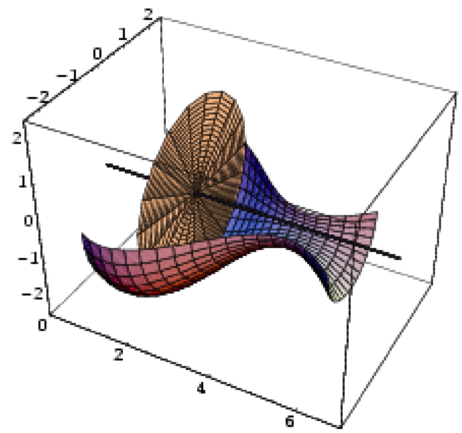
$$\text{Volume} = \int_a^b \pi y^2 \, dx = \int_a^b \pi (f(x))^2 \, dx$$

Here is a picture of the 3-dimensional object.



We justify this by starting with a thin (infinitesimally thin) strip of the region under the graph, from x to $x + dx$ and seeing what happens when we rotate that about the x-axis. This pencil-like thing rotates though a thin disk. The disk is related to the whole object we are considering by being a *slice* through it (sliced perpendicular to the x-axis).





Maybe it would help to think of your object as a (very straight) carrot and then think of it as chopped up into slices. Or think of a French bread sliced up, a perfectly round French bread though, and the slices have to be extremely thin.

Though the radius of our object varies as we go from one end to the other, we are looking at such a thin slide that we can forget any variation in radius. So our thin slice is a circular disk. The radius of the disk is the height $y = f(x)$ of the graph. We can't ignore the thickness of the slice, though it is very small dx because then we would have no volume.

Taking the thickness into account, we should perhaps think of the slice not as a disk but as a very short circular cylinder, radius $r = y = f(x)$ and 'height' or thickness $h = dx$. The formula $\pi r^2 h$ for the volume of a cylinder gives

$$\text{volume of slice} = \pi y^2\,dx = \pi(f(x))^2\,dx$$

When we 'add' up all the volumes of the little slices we end up with the promised formula for the total volume of the object.

7A.15 Example. A spherical cap is made by cutting the top off a solid ball of radius 2. The cut is made at distance 1 from the centre of the ball. What is the volume of the cap?

Solution: We can consider this as a problem fitting the above formula. We should turn the cap so it has the x-axis perpendicular to its flat face and so that the cap is circularly symmetric about the x-axis. (So when you rotate it about the axis, it stays where it was.)

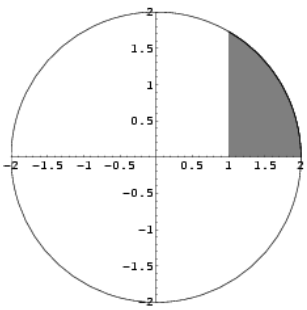
If we take the circle of radius 2 about the origin in the plane, equation $x^2 + y^2 = 2^2$, we can solve for y in terms of x to get $y = \pm\sqrt{4 - x^2}$. The plus sign gives the upper semicircle and our cap shape will emerge if we rotate the area under

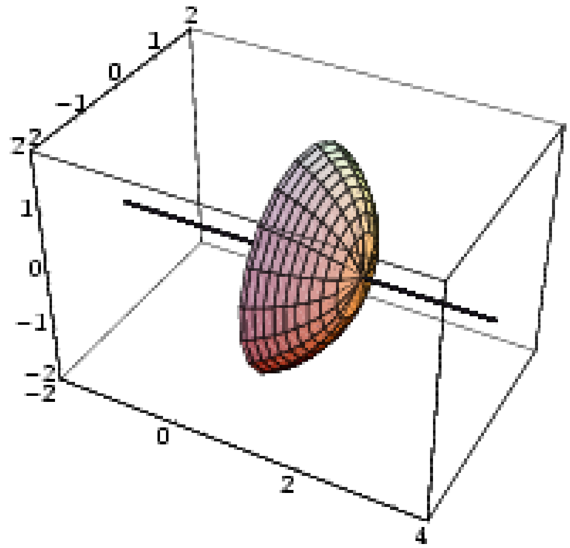
$$y = \sqrt{4 - x^2} \qquad (1 \le x \le 2)$$

about the x-axis. So the volume we want is

$$\int_{x=1}^{2} \pi y^2\,dx = \int_1^2 \pi(4 - x^2)\,dx$$

and this works out to be $\quad \ldots \quad 5\pi/3.$



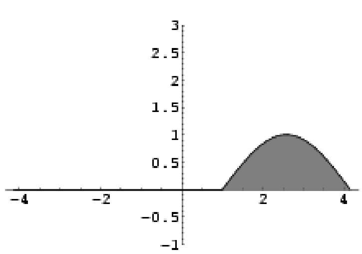


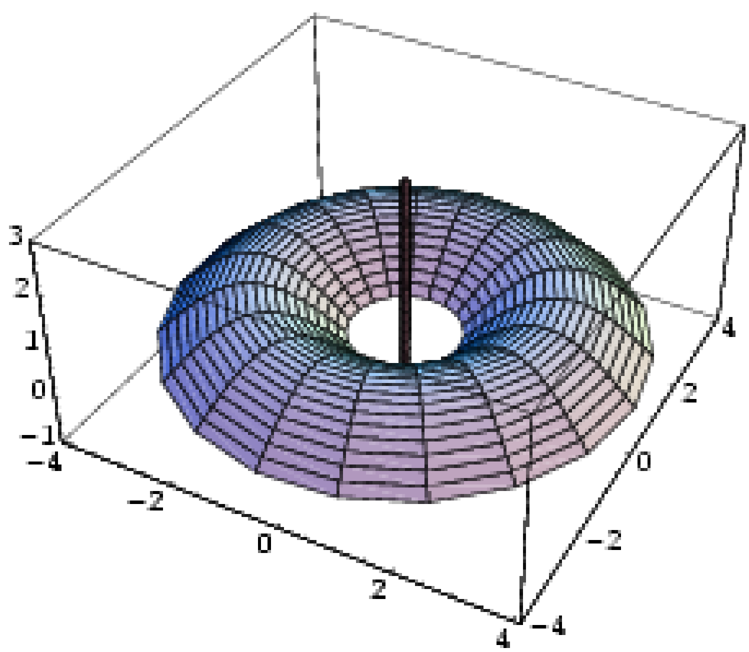
7A.16 Volumes of revolution (method of cylindrical shells). We start again with the area in the plane 'under' a graph $y = f(x)$, $a \leq x \leq b$. Imagine this constructed as a flat piece of cardboard and then imagine rotating it this time about the y-axis. As it rotates around and around it will sweep through a part of space. Our volume formula (for this case of *rotating the area under the graph about the y-axis*) is

$$\text{Volume} = \int_a^b 2\pi x y \, dx = \int_a^b 2\pi x f(x) \, dx$$

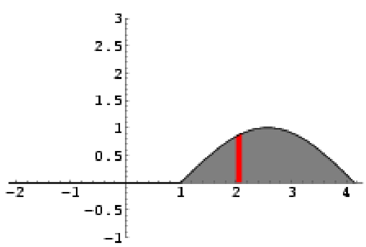
For this formula, we need $a \geq 0$ and $y = f(x) \geq 0$ for all x in the range $a \leq x \leq b$.

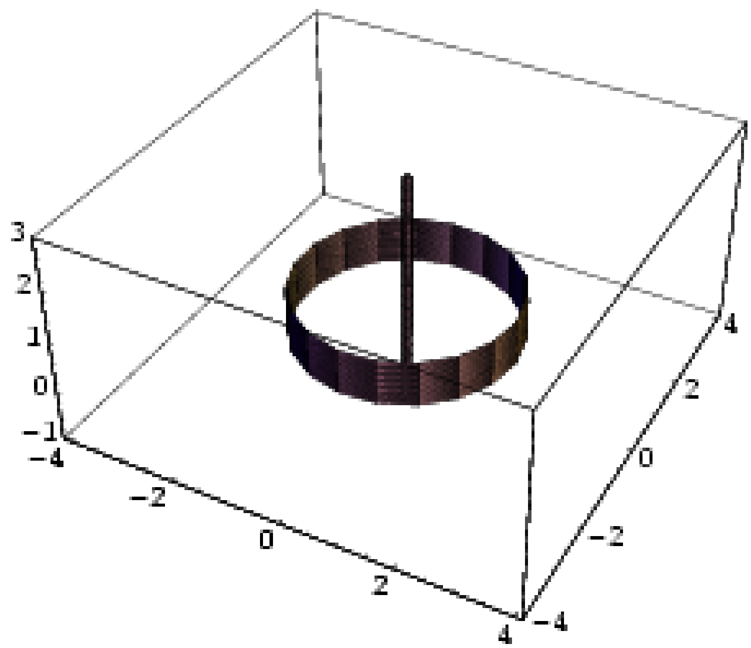
Here is a picture of the sort of thing we have in mind, a graph $y = \sin(x-1)$ for $1 \leq x \leq 1+\pi$ and the shape you get by rotating it about the vertical axis.





We again justify this by starting with a thin (infinitesimally thin) strip of the region under the graph, from x to $x + dx$ and seeing what happens when we rotate that about the y-axis. This pencil-like thing rotates though a thin shell. Or maybe you would like to think of it as a section of pipe. (The wall of the pipe is thin, but not quite zero, so that there is a volume of material needed to make the section of pipe. We are going to 'add' the volumes of these things.) Here is a picture of one.





The shell/pipe has inside radius x, outside radius $x + dx$ and height $y = f(x)$. We get its volume by filling it in to get a solid cylinder and subtracting the volume required to fill it.

Remember that the formula for the volume of a cylinder is $\pi r^2 h$. To fill the thing in we need to add a volume of $\pi x^2 y = \pi x^2 f(x)$ and when we fill it in we will have a solid cylinder of radius $x + dx$, height $y = f(x)$ and so the volume after filling it in is $\pi(x + dx)^2 y = \pi(x + dx)^2 f(x)$. So, subtracting, we get

$$\begin{aligned} \text{volume of shell} &= \pi(x + dx)^2 y - \pi x^2 y \\ &= \pi(x^2 + 2x dx + (dx)^2 - x^2) y \\ &= 2\pi xy\, dx + \pi y (dx)^2 \end{aligned}$$

I don't want the $\pi y(dx)^2$ term and so I'm going to ignore it. That is not so bad a thing to do because it is really very small. dx is already infinitesimal (= unbelievably small) and $(dx)^2$ is an unbelievably small fraction of something that is already unbelievably small. Perhaps a more convincing argument uses limits instead of infinitesimals. If we did things with Riemann sums, we would have small finite widths Δx instead of infinitesimals dx. Then we would need some fairly large number n steps of size Δx to get from $x = a$ to $x = b$. In fact $\Delta x = \dfrac{b - a}{n}$. When we add up the n different quantities for the volumes of the n slices, the sum of the n quantities

$$\pi y(\Delta x)^2 = \pi y \left(\frac{b-a}{n}\right)^2$$

will still have an n in the denominator and so their contribution will $\to 0$ as $n \to \infty$.

In any case, we can ignore the $\pi y (dx)^2$ bit and add up the rest to get

$$\int_{x=a}^{b} 2\pi x y \, dx$$

We would not be doing this right if $x < 0$ ever happened. We took x as a radius and so it needed to be positive. Also we really needed $y = f(x) =$ a height to be positive.

7A.17 Example. For the same graph as in the previous example

$$y = \sqrt{4 - x^2} \qquad (1 \le x \le 2)$$

find the volume swept out when we rotate the region under this graph about the (vertical) y-axis.
Solution: This is all set up to use the formula and we get

$$\text{Volume} = \int_{x=1}^{2} 2\pi x y \, dx = \int_{x=1}^{2} 2\pi x \sqrt{4 - x^2} \, dx$$

and after some work ... this turns out to be $2\pi\sqrt{3}$.

Here is a picture of the object. The picture is cut away to help see what it looks like. (If you took a half ball of radius 2 and drilled a hole of radius 1 out of it, this is what would be left.)