Chapter 6: Integration: partial fractions and improper integrals

Course 1S3, 2006–07

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These are just summaries of the lecture notes, and few details are included. Most of what we include here is to be found in more detail in Anton.

6.1 Remark. Here is a quick reminder of the basics of integration, before we move on to partial fractions.

6.2 Remark. Unlike differentiation where we can differentiate almost anything we can write down using the basic rules (including the chain rule, product rule and quotient rule), with integration it is easy to come across simple-looking things we will not be able to do. One example is $\int \sin x^2 dx$.

For each differentiation formula, we have a corresponding integration formula.

| Derivative formula | Integration formula |
|--------------------------------------|---|
| $\frac{d}{dx}x^n = nx^{n-1}$ | $\int x^{n} dx = \frac{1}{n+1} x^{n+1} + C \text{ if } n \neq -1$ |
| $\frac{d}{dx}e^x = e^x$ | $\int e^x dx = e^x + C$ |
| $\frac{d}{dx}\sin x = \cos x$ | $\int \cos x dx = \sin x + C$ |
| $\frac{d}{dx}\cos x = -\sin x$ | $\int \sin x dx = -\cos x + C$ |
| $\frac{d}{dx}\tan x = \sec^2 x$ | $\int \sec^2 x dx = \tan x + C$ |
| $\frac{d}{dx}\sec x = \sec x \tan x$ | $\int \sec x \tan x dx = \sec x + C$ |

We already see a gap and we need $\int \frac{1}{x} dx = \ln |x| + C$. One way to define the natural logarithm is

$$\ln x = \int_1^x \frac{1}{t} dt \qquad (x > 0)$$

and then the fact that its derivative is 1/x (for x > 0) is an immediate consequence of the Fundamental theorem. Essentially the antiderivative of 1/x is new thing (much more complicated

than 1/x) and there is almost no limit to the new functions we can come across this way — new in the sense that they cannot be written in terms of familiar functions. The natural logarithm function is a useful thing in many ways.

Apart from these basic integrals, there are integration formulae that follow from the chain rule for differentiation and the product rule for differentiation. Essentially, when we rearrange the integral of the two formulae, we get the method of substitution and integration by parts.

Aside from these there are no real 'methods' but there is a compendium of what are essentially 'tricks' to make the methods useful in given situations. Included amongst these are ways of integrating (many) combinations of trigonometric functions (evaluated at x, or maybe ax for a constant a rather than at more complicated expressions) and inverse trigonometric substitutions. These latter have the effect of introducing trigonometry into integration problems that seem to be unrelated to trigonometry.

A lot of this was covered in 1S1. If in doubt look in Anton, or at the appendix to this part of the notes. Not all of the trigonometric and inverse trigonometric (and inverse hyperbolic) substitutions were done in 1S1 and so we fill in the gaps here.

6.3 Trigonometric Integrals. For the first 3 items we are just listing them for context. (They were covered in 1S1 and also you can find more detail in the appendix below.)

(i) Powers of $\sin x$ times powers of $\sin x$ with one power odd

Method: For

$$\int \sin^n x \cos^m x \, dx$$

- if n = the power of $\sin x$ is odd, substitute $u = \cos x$
- if m = the power of $\cos x$ is odd, substitute $u = \sin x$
- (ii) Powers of $\sin x$ times powers of $\sin x$ with both powers even

Method: use the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

(iii) **Powers of** $\sin x$ and $\cos x$

Method: Use the previous two methods treating

$$\int \sin^n x \, dx = \int \sin^n x (\cos x)^0 \, dx$$

and similarly for $\int \cos^m x \, dx$ (that is treat the second power as the zeroth power).

There is another way to do these problems, and it is given in the book by Anton. Possibly also some of you know it already.

Alternative method: Use the reduction formulae

$$\int \sin^{n} x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$
$$\int \cos^{m} x \, dx = \frac{1}{m} \cos^{m-1} x \sin x + \frac{m-1}{m} \int \cos^{m-2} x \, dx$$

(valid for $n \ge 2$ and $m \ge 2$) to express the integrals of a power in terms of integrals where the power is reduced by 2. Using the formula enough times, will get down to power 1 or zero eventually.

- (iv) **Powers of** $\tan x$
 - The first power $\int \tan x \, dx$

We can work this out using the odd powers of $\sin x$ method.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x \, du = -\sin x \, dx$

$$= \int \frac{\sin x}{u} \frac{du}{-\sin x}$$

$$= \int -\frac{1}{u} \, du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

Usually this is rewritten using

$$-\ln|\cos x| = \ln \frac{1}{|\cos x|} = \ln \left|\frac{1}{\cos x}\right| = \ln|\sec x|$$

and the result is

$$\int \tan x \, dx = \ln |\sec x| + C$$

• The second power $\int \tan^2 x \, dx$

This is really quite easy if we recall the identity $1 + \tan^2 x = \sec^2 x$. We get

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$$

• Higher powers $\int \tan^n x \, dx \ (n \ge 3)$

What we do here is to split off two powers (that is $tan^2 x = \sec^2 x - 1$)

$$\int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$$
$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$
$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$
In the first integral, substitute
$$u = \tan x$$

$$du = \sec^{2} x \, dx$$

$$= \int u^{n-2} \, du - \int \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} u^{n-1} - \int \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

This is a reduction method. It expresses $\int \tan^n x \, dx$ in terms of a similar integral with the power reduced by 2. For example

$$\int \tan^5 x \, dx = \frac{1}{5-1} \tan^{5-1} x - \int \tan^{5-2} x \, dx$$
$$= \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx$$
$$= \frac{1}{4} \tan^4 x - \left(\frac{1}{3} \tan^2 x - \int \tan x \, dx\right)$$

and we know $\int \tan x \, dx$.

- (v) **Powers of** $\sec x$
 - The first power: $\int \sec x \, dx$

There is no really nice way to find this out. What we do is write down the answer

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

and show that it works. That is we check that the right hand side is an antiderivative for $\sec x$:

$$\frac{d}{dx}\ln|\sec x + \tan x| = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \sec x$$

• The second power: $\int \sec^2 x \, dx$

This is a basic formula

$$\int \sec^2 x \, dx = \tan x + C$$

• *Higher powers:* $\int \sec^n x \, dx \, (n \ge 3)$

What we do here is split off two powers ($\sec^2 x$) and use integration by parts

$$\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$$

$$\text{Let} \quad u = \sec^{n-2} x \qquad dv = \sec^2 x \, dx$$

$$du = (n-2) \sec^{n-3} x \sec x \tan x \, dx \qquad v = \tan x$$

$$= \int u \, dv$$

$$= uv - \int v \, du$$

$$= \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

We can solve this equation for our 'unknown' integral $\int \sec^n x \, dx$

$$(n-1)\int \sec^{n} x \, dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x \, dx$$
$$\int \sec^{n} x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

This is a reduction formula and can be used as the other reduction formulae were used.

6.4 Recap about inverse trigonometric and hyperbolic functions. In 1S1 you learned about inverse functions. Here we will review the basic ideas briefly and spell out some details about inverses of trigonometric and hyperbolic functions.

Recall that the inverse of a function y = f(x) is a rule that finds what x must be when you know y. That is, the inverse function applied to y, which we write $f^{-1}(y)$, is the value of x with f(x) = y. So $f^{-1}(y) = x$ should be equivalent to y = f(x).

Another way to say it is that the inverse function should exactly undo whatever the function does (thinking of f(x) as the rule f doing something to x).

A snag that arises is that there is often no way to find x just knowing y. For example, in the simple case $y = f(x) = x^2$, we cannot find x exactly if we know y = 4. The equation f(x) = 4 (or $x^2 = 4$) has two solutions x = 2 and x = -2. Since a function can only have one value, there is no good way to define $f^{-1}(4)$. What we always do in this example is to define \sqrt{y} (for $y \ge 0$) to be the positive square root of y, so that $\sqrt{4} = 2$ (not -2). The square root function is a function that partly undoes what the function $f(x) = x^2$ does. It is true that $(\sqrt{y})^2 = y$, but $\sqrt{x^2} = |x|$ is the same as x only when $x \ge 0$.

What we can do is consider a (somewhat artificially) restricted version of the function $y = f(x) = x^2$ where we allow only $x \ge 0$. Also the values in this case are $y \ge 0$ and we can say we are considering the function

$$\begin{array}{rcl} f \colon [0,\infty) & \to & [0,\infty) \\ f(x) & = & x^2 \end{array}$$

where the target set (or co-domain set) on the right is adjusted to be equal to the range. Then we will be able to find x unambiguously from knowing y = f(x). It will be right then that the inverse function

$$\begin{aligned} f^{-1} \colon [0,\infty) &\to & [0,\infty) \\ f^{-1}(y) &= & \sqrt{y}. \end{aligned}$$

The graph of $y = f^{-1}(x)$ is sort of the same as the graph $x = f^{-1}(y)$ — but it is not really the same because the axes are swopped. The graph $x = f^{-1}(y)$ is really identical with the graph y = f(x) but the graph $y = f^{-1}(x)$ is related to it by taking a reflection in the line y = x.







From the picture and the fact that the tangent line to the graph y = f(x) becomes the tangent line to the graph $y = f^{-1}(x)$ of the inverse function when you reflect in the line y = x, it is not hard to see that the slope of one tangent line is the reciprocal of the slope of the other. Since the slope is got by $m = \frac{y_2 - y_1}{x_2 - x_2}$ for two points (x_1, y_1) and (x_2, y_2) on the line, you can see that interchanging x and y (which is what happens when you reflect in the line y = x) turns the fraction for the slope upside down.

The Liebniz $\frac{dy}{dx}$ notation for derivatives also suggests the same fact. If we write y = f(x) and $x = f^{-1}(y)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = (f^{-1})'(y)$. There is a theorem that if $\frac{dy}{dx} = f'(x)$ exists and is not zero then the dreivative of the inverse function will exists and will be given by

$$(f^{-1})'(y) = \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

In the case of the $f(x) = x^2$ (with $x \ge 0$) we have $f'(x) = 2x \ne 0$ as long as we avoid x = 0. So the derivative of the inverse function $f^{-1}(y) = \sqrt{y}$ is

$$(f^{-1})'(y) = \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{2x}$$

The only snag with this is that we have got

$$\frac{d}{dy}\sqrt{y} = \frac{1}{2x}$$

and usually we want $\frac{d}{dy}\sqrt{y}$ in terms of y, not x. But we have $x = \sqrt{y} = f^{-1}(y)$ here and so we get the expected formula

$$\frac{d}{dy}\sqrt{y} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$

Finally, there is one more trick. The $y = f(x) = x^2$ and $x = f^{-1}(y) = \sqrt{y}$ notation is handy for a while, but eventually we would want to consider the square root function as a thing on its own, not necessarily thinking about how it relates to the function $f(x) = x^2$. That is, we might look at $g(x) = \sqrt{x}$ for $x \ge 0$. We can change y to x throughout

$$\frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

and get

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

We now run through the inverse trig functions briefly, using the same approach as what we had above for the square root example.

sin⁻¹: The function $x = \sin \theta$ certainly does not have an inverse because $\sin \theta = \sin(\theta + 2\pi) = \sin(\pi - \theta)$ and so knowing only $x = \sin \theta$ we cannot know θ .

To get an inverse function, by convention we artificially restrict θ to lie in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and consider the function

$$f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \left[-1, 1\right]$$
$$x = f(\theta) = \sin \theta$$

The inverse function of this does make sense and is sometimes called $\arcsin x$ to emphasise that is is not the inverse of the ordinary $\sin \theta$ function, rather a restricted version of $\sin \theta$. However we will write is as $\sin^{-1} x$ (another convention that is used at least as often).¹

In summary, $\sin^{-1} x$ only makes sense if $-1 \le x \le 1$ and

$$\sin^{-1} x = \theta$$
 means $x = \sin \theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

To work out the derivative, we have for $x = \sin \theta$,

$$\frac{dx}{d\theta} = \cos\theta$$

and so

$$\frac{d\theta}{dx} = \frac{d}{dx}\sin^{-1}x = \frac{1}{\cos\theta}$$

To get the answer in terms of x we use $\cos^2 \theta + \sin^2 \theta = 1$, hence $\cos^2 \theta = 1 - \sin^2 \theta = 1 - x^2$. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ we have $\cos \theta \ge 0$ and so $\cos \theta = \sqrt{1 - x^2}$. Thus we get finally

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

¹Mathematica uses ArcSin[x] for this function.

(We have to rule out x = 1 and x = -1 to avoid division by zero.)

The integration version of this is

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}x + C$$



 $\tan^{-1} x$: Since $\tan(\theta + \pi) = \tan \theta$ we have a similar problem with finding θ knowing $x = \tan \theta$ and we 'solve' the problem by restricting θ to be in the range $-\pi/2 < \theta < \pi/2$.

That is, we consider the function

$$f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$
$$x = f(\theta) = \tan \theta$$

The inverse function of this does make sense and is sometimes called $\arctan x$ but we will use the alternative $\tan^{-1} x$. In summary

$$\tan^{-1} x = \theta$$
 means $x = \tan \theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Here are the graphs of $y = \tan \theta \ (-\pi/2 < x < \pi/2)$ and $y = \tan^{-1} x$ 1 2 0.5 -1.5 0.5 -4 1.5 $^{-2}$ 2 4 -1-0-5 1 -0/5 -2-1-4

We can work out the derivative

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

in a similar way to the method used for the derivative of $\sin^{-1} x$. The corresponding integration formula is

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C.$$

 $\cosh^{-1} x$: Recall that $\cosh x = (e^x - e^{-x})/2$. (We introduced it in the appendix to Chapter 2, in connection with parametric plotting of a hyperbola.)

We could point out that, now we know $\frac{d}{dx}e^x = e^x$ from 1S1, we can say

$$\frac{d}{dt}\cosh t = \frac{d}{dt}\frac{e^t + e^{-t}}{2} = \frac{e^t - e^{-t}}{2} = \sinh t$$

(This differs by a sign from $\frac{d}{d\theta}\cos\theta = -\sin\theta$. It is easy to check that $\frac{d}{dt}\sinh t = \cosh t$, which looks the same as the familiar $\frac{d}{d\theta}\sin\theta = \cos\theta$.)

Also $\cosh(-t) = \cosh t$ is easy to see and so we cannot hope to find t knowing only $x = \cosh t$. However, if we restrict to $t \ge 0$ we can solve for t in terms of x as long as x is in the range of $\cosh t$ (which means $x \ge 1$). One way to see $\cosh t \ge 1$ always is to notice $\cosh^2 t - \sinh^2 t = 1$ so that $\cosh^2 t \ge 1$ must be true.

So $\cosh^{-1} x$ only makes sense if $x \ge 1$ and

$$t = \cosh^{-1} x$$
 means $\cosh t = x$ and $t \ge 0$.



We can work out the derivative

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2 - 1}}$$

in a similar way to the method used for the derivative of $\sin^{-1} x$ and $\tan^{-1} x$. The corresponding integration formula is

$$\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \cosh^{-1} x + C.$$

A new feature of the function $\cosh^{-1} x$ is that we can write a formula for it using the natural logarithm. (Maybe this is not totally surprising since $\cosh t$ is described by exponentials.) The idea is that we can solve

$$x = \cosh t = \frac{e^t + e^{-t}}{2} = \frac{e^t + \frac{1}{e^t}}{2}$$

for e^t as follows:

$$2x = e^{t} + \frac{1}{e^{t}}$$

$$2xe^{t} = (e^{t})^{2} + 1$$

$$0 = (e^{t})^{2} - 2xe^{t} + 1$$

$$(e^{t})^{2} - 2xe^{t} + 1 = 0$$

This is a quadratic equation for e^t and so

$$e^t = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

Since $e^t \ge e^0 = 1$ and $x \ge 1$ we must have the plus sign

$$e^t = -x + \sqrt{x^2 - 1}$$

and this says

$$t = \ln(x + \sqrt{x^2 - 1})$$

So we end up with

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

 $\sinh^{-1} x$: The function $y = \sinh t$ is genuinely invertible (no need to introduce any restrictions on the domain) and the range is all of \mathbb{R} . So

$$t = \sinh^{-1} x$$
 just means exactly $\sinh t = x$



We can work out the derivative

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2 + 1}}$$

in a similar way to the method used previously. The corresponding integration formula is

$$\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \sinh^{-1} x + C.$$

We can also solve $x = \sinh t$ for t in terms of x in a way rather similar to that used for $\cosh t$ above and get

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

6.5 Inverse trigonometric substitutions. We now consider a class of substitutions which seem quite counter intuitive. They are related to the derivative formulae for the inverse trigonometric and inverse hyperbolic functions.

Here are the derivative formulae from above summarised:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$$

The corresponding substitution methods are:

• integrals involving $\sqrt{1-x^2}$, substitute $x = \sin \theta$.

More generally, integrals involving $\sqrt{a^2 - u^2}$ (with a constant) — try substituting $u = a \sin \theta$

- integrals involving $\frac{1}{a^2 + u^2}$ try substituting $u = a \tan \theta$
- integrals involving $\sqrt{u^2 a^2}$ try substituting $u = a \cosh t$
- integrals involving $\sqrt{u^2 + a^2}$ try substituting $u = a \sinh t$

Examples:

(i)
$$\int \frac{1}{\sqrt{4-x^2}} dx$$

Solution: According the method described (since $\sqrt{4-x^2} = \sqrt{2^2 - x^2} = \sqrt{a^2 - u^2}$ with a = 2 and u = x) we should try substituting

$$u = a \sin \theta$$
 that is $x = 2 \sin \theta$

Notice that this looks backwards compared to other substitutions we have done. x is the orignal variable and θ is the new one but we are expressing the old in terms of the new. Normally we express the new in terms of the old. Indeed, this is what is happening here, but we have not expressed it totally clearly. We can rewrite $x = 2 \sin \theta$ as

$$\frac{x}{2} = \sin \theta$$

and the way to say what θ is in a way that is not ambiguous, or to say what we really intend, is to have

$$\theta = \sin^{-1}\left(\frac{x}{2}\right)$$

Nevertheless, the way we started is very convenient for substitution. We have

$$x = 2\sin\theta$$
$$dx = 2\cos\theta d\theta$$
$$\int \frac{1}{\sqrt{4 - x^2}} dx = \int \frac{1}{\sqrt{4 - (2\sin\theta)^2}} 2\cos\theta d\theta$$
$$= \int \frac{1}{\sqrt{4 - 4\sin^2\theta}} 2\cos\theta d\theta$$
$$= \int \frac{1}{\sqrt{4(1 - \sin^2\theta)}} 2\cos\theta d\theta$$
$$= \int \frac{1}{\sqrt{4\cos^2\theta}} 2\cos\theta d\theta$$
$$= \int \frac{1}{2\cos\theta} 2\cos\theta d\theta$$
$$= \int 1 d\theta$$

The point of the substitution (and the reason it is often advantageous) is that we get rid of the square root. In this example, everything is very simple, but getting rid of the square root is the essential thing.

We get

$$\int \frac{1}{\sqrt{4-x^2}} \, dx = \theta + C = \sin^{-1}(x/2) + C$$

using the earlier expression for θ with the inverse function.

(ii)
$$\int \sqrt{9 - 4(x+2)^2} \, dx$$

Solution: In this case, we have $\sqrt{a^2 - u^2}$ with $a^2 = 9$, a = 3, u = 2(x + 2) and so our method says to try $u = a \sin \theta$ or

$$2(x+2) = 3\sin\theta$$

$$2 dx = 3\cos\theta d\theta$$

$$dx = \frac{3\cos\theta}{2} d\theta$$

$$\int \sqrt{9 - 4(x+2)^2} dx = \int \sqrt{9 - 9\sin^2\theta} \frac{3\cos\theta}{2} d\theta$$

$$= \int \sqrt{9\cos^2\theta} \frac{3\cos\theta}{2} d\theta$$

$$= \int 3\cos\theta \frac{3\cos\theta}{2} d\theta$$

$$= \frac{9}{2} \int \cos^2\theta d\theta$$

$$= \frac{9}{4} \int 1 + \cos 2\theta d\theta$$

$$= \frac{9}{4} (\theta + \frac{1}{2}\sin 2\theta) + C$$

To get the answer in terms of x, we need θ in terms of x

$$2(x+2) = 3\sin\theta$$

$$\frac{2}{3}(x+2) = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{2}{3}(x+2)\right)$$

and we could get a correct answer by replacing θ by this everywhere in the answer above. However, there is a tidier way to deal with $\sin 2\theta$ using

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{4}{3}(x+2)\cos\theta$$

and we can express $\cos \theta$ in terms of $\sin \theta$ by looking at a right angled triangle with θ as an angle, 2(x + 2) as the length of the opposite side and 3 as the length of the hypotenuse. Then Pythagoras' theorem gives the adjacent side as

$$\sqrt{\text{hypoteneuse}^2 - \text{opposite}^2} = \sqrt{9 - 2(x+2)^2}$$

and so

$$\cos \theta = \frac{\text{adjacent}}{\text{hypoteneuse}} = \frac{\sqrt{9 - 4(x+2)^2}}{3}$$

We could also get to this from $\cos \theta = \sqrt{1 - \sin^2 \theta}$.

In any case we end up with a (reasonably tidy) answer

$$\int \sqrt{9 - 4(x+2)^2} \, dx = \frac{9}{4}\theta + \frac{9}{8}\sin 2\theta + C$$

= $\frac{9}{4}\sin^{-1}\left(\frac{2}{3}(x+2)\right)$
 $+\frac{9}{8}\frac{4}{3}(x+2)\frac{\sqrt{9 - 4(x+2)^2}}{3} + C$
= $\frac{9}{4}\sin^{-1}\left(\frac{2}{3}(x+2)\right) + \frac{1}{2}(x+2)\sqrt{9 - 4(x+2)^2} + C$

(iii)
$$\int \sqrt{-4x^2 - 16x - 7} \, dx$$

Solution: For quadratics inside a square root like this, what we should do first is complete the square. That is, rearrange the x^2 and x terms so that (with a suitable constant) we get a multiple of a perfect square

$$-4x^{2} - 16x - 7 = -4(x^{2} + 4x) - 7$$

= $-4(x^{2} + 4x + 4 - 4) - 7$
= $-4(x^{2} + 4x + 4) + 9$
= $9 - 4(x + 2)^{2}$

This means that not only is this problem similar to the previous one, it is in fact the same problem again (now that we completed the square).

6.6 Partial Fractions. Partial fractions are a technique from algebra, but our reason for dealing with them is that they can in principle help find every integral of the form

$$\int \frac{p(x)}{q(x)} \, dx$$

where p(x) and q(x) are polynomials.

Except in a few special cases, we don't yet know how to find such integrals. One special case, where we don't need partial fractions, is where q'(x) = p(x) or $q'(x) = \alpha p(x)$ for some constant α , because in these cases we can make a substitution u = q(x), du = q'(x) dx and it will work out nicely. In fact substitution would also work if $q(x) = r(x)^n$ for some $n \ge 1$ and $r'(x) = \alpha p(x)$ for a constant α —we can substitute u = r(x), $du = r'(x) dx = \alpha p(x) dx$,

$$\int \frac{p(x)}{q(x)} dx = \int \frac{p(x)}{r(x)^n} dx = \int \frac{p(x)}{u^n} \frac{du}{\alpha p(x)} = \int (1/\alpha) \frac{1}{u^n} du$$

The idea of partial fractions is to rewrite $\frac{p(x)}{q(x)}$ as a sum of fractions with simple denominators and numerators that are somehow small compared to the denominator. We need to explain exactly how it goes.

To try and get the idea, let us first look at what we can do with ordinary numerical fractions. Consider $\frac{98}{45}$. What we can see first is that it is not a proper fraction. 45 divides into 98 2 times with a remainder 8. So we can write

$$\frac{98}{45} = 2 + \frac{8}{45} = 2\frac{8}{45}$$

and the 2 part is not a fraction at all. We concentrate on the proper faction part 8/45. By 'simple denominator' in the case of numerical fractions we mean denominator that is a power of a *prime number*. And by denominator that is 'small compared to the denominator', we mean smaller in absolute value than the prime number that occurs in the denominator.

For example, in the case of 8/45, the denominator factors $45 = 5 \times 9 = 5 \times 3 \times 3 = 5 \times 3^2$, and the prime powers that are possibly needed in this case are 5, 3 and 3^2 . We should be able to write any proper fraction with denominator 45 as a sum

$$\frac{a}{5} + \frac{b}{3} + \frac{c}{3^2}$$

where |a| < 5, |b| < 3 and |c| < 3. In our example,

$$\frac{8}{45} = \frac{2}{5} + \frac{0}{3} + \frac{-2}{3^2}$$

Other examples are

$$\frac{34}{45} = \frac{1}{5} + \frac{1}{3} + \frac{2}{3^2}$$

and

$$\frac{12}{45} = \frac{4}{15} = \frac{3}{5} + \frac{-1}{3} = \frac{3}{5} + \frac{-1}{3} + \frac{0}{3^2}$$

(In the last example there is no need for a denominator 3^2 because the fraction 12/45 = 4/15 in lowest terms.)

One can write all proper numerical fractions in this way. You could maybe experiment with fractions n/20 if you like, but we are not going to need to do this. It is just here as a sort of motivation for doing something similar for fractions of polynomials p(x)/q(x)

We need to talk about factoring polynomials and about what would be the corresponding thing to a prime number for polynomials.

To start with, a polynomial is an expression you get by taking a finite number of powers of x with constant coefficients in front and adding them up. For example

$$p(x) = 4x^2 - x + 17$$

or

$$p(x) = 27x^{11} + 15x^{10} - x^9 + x^8 + 11x^2 + 5$$

are polynomials. The highest power of x that has a nonzero coefficient in front is called the *degree* of the polynomial. The examples above have degree 2 and degree 11.

What is handy to know is that when we multiply polynomials, the degrees add. So $(x + 1)(x + 5)(x^2 + x + 11)$ will have degree 1 + 1 + 2 = 4 when it is multiplied out. Constant polynomials have degree 0, except the zero polynomial — we are best not giving any degree to the zero polynomial.

Now, what kind of polynomial corresponds to a prime number? Prime numbers are whole (positive) numbers that cannot be factored except in a very silly way, as themselves times 1 (or minus themselves times -1). So the only factorisations of 5 are $5 = 5 \times 1 = (-5) \times (-1)$. We do not include 1 as a prime number. With polynomials, we consider a factorisation to be genuine if the factors each have degree at least 1. So $2x^2 + 8x + 2 = (2x + 4)(x + 2)$ is a genuine factorisation, but 2x + 4 = 2(x + 2) is not considered genuine, nor is 2x + 1 = 2(x + 1/2). The factors must contain x to at least the first power. Anything of degree 1 certainly cannot be factored then. Some things of degree 2 can be factored, such as

$$x^2 + 5x + 4 = (x+1)(x+4)$$

but other quadratics cannot be factored if we don't allow complex numbers to be used. We *cannot* factor

$$x^{2} + 2x + 2 = (x - \alpha)(x - \beta)$$

because if we could then the roots of $x^2 + 2x + 2$ would be α and β . The roots of $x^2 + 2x + 2$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

and these are complex numbers. So α and β would have to be these complex numbers.

A remarkable fact is that every polynomial of degree 3 or more **can** be factored, at least in theory. It does not mean it is easy to find the factors, unfortunately. What you can sometimes rely on to factor polynomials is the **Remainder Theorem.** It says that if p(x) is a polynomial and you know a root x = a (that is a value a so that p(a) = 0), then x - a must divide p(x).

Using the theory, just as in principle whole numbers can be factored as a product of prime numbers, so polynomials p(x) with real coefficients can be factored as a product of linear factors like x - a and quadratic factors $x^2 + bx + c$ with complex roots. If the coefficient of the highest power of x in p(x) is not 1, then we also need to include that coefficient. So a complete factorisation of

$$2x^{2} + 8x + 2 = (2x + 4)(x + 2) = 2(x + 2)(x + 2) = 2(x + 2)^{2}.$$

For $3x^3 + 3x^2 + 6x + 6 = 3(x^3 + x^2 + 2x + 2)$, you can check that x = -1 is a root and so x - (-1) = x + 1 must divide it. We get

$$3x^3 + 3x^2 + 6x + 6 = 3(x+1)(x^2+2)$$

from long division.

Now we can outline **how partial fractions works** for a fraction $\frac{p(x)}{q(x)}$ of two polynomials:

Step 0: (preparatory step). If the degree of the numerator p(x) is not strictly smaller than the degree of the denominator q(x), use long division to divide q(x) into p(x) and obtain a quotient Q(x) and remainder R(x). Then

$$\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$$

and degree(R(x)) < degree(q(x)).

We concentrate then on the 'proper fraction' part R(x)/q(x).

Step 1: Now factor q(x) completely into a product of linear factors x - a and quadratic factors $x^2 + bx + c$ with complex roots.

Gather up any repeated terms, so that if (say) $q(x) = (x - 1)(x + 2)(x^2 + 3)(x + 2)$ we should write it as $q(x) = (x - 1)(x + 2)^2(x^2 + 3)$.

Step 2: Then the proper fraction $\frac{R(x)}{q(x)}$ can be written as a sum of fractions of the following types:

(i)
$$\frac{A}{(x-a)^m}$$

(ii)
$$\frac{Bx+C}{(x^2+bx+c)^k}$$

where we include all possible powers $(x - a)^m$ and $(x^2 + bx + c)^k$ that divide q(x). The A, B, C stand for constants.

As examples, consider the following. We just write down what the partial fractions look like. In each case, we start with a proper fraction where the denominator is completely factored already. So some of the hard work is already done.

(i)
$$\frac{x^2 + x + 5}{(x-1)(x-2)(x-3)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}$$

(ii)
$$\frac{x^3 + 2x + 7}{(x+1)^2(x-4)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{x-4}$$

(iii)
$$\frac{x^2 + x + 11}{(x+1)(x^2 + 2x + 2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2 + 2x + 2}$$

(iv)
$$\frac{x^4 + x + 11}{(x+1)(x^2 + 2x + 2)^2} = \frac{A}{x+1} + \frac{B_1 x + C_1}{x^2 + 2x + 2} + \frac{B_2 x + C_2}{(x^2 + 2x + 2)^2}$$

To make use of these, we have to be able to find the numbers A, B, C, ... that make the equation true. Take the first example

$$\frac{x^2 + x + 5}{(x-1)(x-2)(x-3)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}.$$

To find the appropriate A_1, A_2, A_3 , we multiply across by the original denominator (x - 1)(x - 2)(x - 3). This has the effect of clearing all the fractions.

$$x^{2} + x + 5 = \frac{A_{1}}{x - 1}(x - 1)(x - 2)(x - 3) + \frac{A_{2}}{x - 2}(x - 1)(x - 2)(x - 3) + \frac{A_{3}}{x - 3}(x - 1)(x - 2)(x - 3)$$

= $A_{1}(x - 2)(x - 3) + A_{2}(x - 1)(x - 3) + A_{3}(x - 1)(x - 2)$

There are two avenues to pursue from here. In this case, method 1 seems easier to me, but in general method 2 can be as good.

Method 1. Plug in the values of x that make the original denominator (x-1)(x-2)(x-3) = 0.

$$x = 1:$$

$$1 + 1 + 5 = A_{1}(1 - 2)(1 - 3) + A_{2}(0) + A_{3}(0)$$

$$7 = 2A_{1}$$

$$A_{1} = 7/2$$

$$x = 2:$$

$$11 = A_{1}(0) + A_{2}(1)(-1) + A_{3}(0)$$

$$= -A_{2}$$

$$A_{2} = -11$$

$$x = 3:$$

$$17 = 0 + 0 + A_{3}(2)(1)$$

$$A_{3} = 17/2$$

So we get

$$\frac{x^2 + x + 5}{(x-1)(x-2)(x-3)} = \frac{7/2}{x-1} + \frac{-11}{x-2} + \frac{17/2}{x-3}$$

Our interest in this is for integration. We can now easily integrate

$$\int \frac{x^2 + x + 5}{(x - 1)(x - 2)(x - 3)} \, dx = \int \frac{7/2}{x - 1} + \frac{-11}{x - 2} + \frac{17/2}{x - 3} \, dx$$
$$= \frac{7}{2} \ln|x - 1| - 11 \ln|x - 2| + \frac{17}{2} \ln|x - 3| + C$$

Method 2. Multiply out the right hand side.

$$x^{2} + x + 5 = A_{1}(x - 2)(x - 3) + A_{2}(x - 1)(x - 3) + A_{3}(x - 1)(x - 2)$$

= $A_{1}(x^{2} - 5x + 6) + A_{2}(x^{2} - 4x + 3) + A_{3}(x^{2} - 2x + 2)$
= $(A_{1} + A_{2} + A_{3})x^{2} + (-5A_{1} - 4A_{2} - 2A_{3})x + (6A_{1} + 3A_{2} + 2A_{2})$

and compare the coefficients on both sides to get a system of linear equations

These can be solved (by Gaussian elimination, for example) to find A_1, A_2, A_3 .

Method 1 is certainly magic in this case, but there are examples where Method 1 does not get all the unknown so easily.

Another example. Here is one of our previous examples with the numbers worked out.

$$\frac{x^2 + x + 11}{(x+1)(x^2 + 2x + 2)} = \frac{11}{x+1} + \frac{-10x - 11}{x^2 + 2x + 2}$$

To find the integral of this,

$$\int \frac{x^2 + x + 11}{(x+1)(x^2 + 2x + 2)} \, dx = \int \frac{11}{x+1} \, dx - \int \frac{10x+11}{x^2 + 2x + 2} \, dx$$
$$= 11 \ln|x+1| - \int \frac{10x+11}{x^2 + 2x + 2} \, dx$$

To work out the remaining integral, we use the method of completing the square $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x + 1)^2 + 1$ and we can then make the substitution $x + 1 = \tan \theta$. However, there is a trick that saves a lot of simplifying later. The substitution $u = x^2 + 2x + 2$, du = (2x + 2) dx = 2(x + 1) dx would work if the numerator was a multiple of x + 1. What we can do is write

$$\int \frac{10x+11}{x^2+2x+2} \, dx = \int \frac{10x+10}{x^2+2x+2} \, dx + \int \frac{1}{x^2+2x+2} \, dx$$

and make the $u = x^2 + 2x + 2$ substitution in the first half, but the $x + 1 = \tan \theta$, $dx = \sec^2 \theta \, d\theta$ in the second. We get

$$\int \frac{10x+11}{x^2+2x+2} dx = \int \frac{10x+10}{u} \frac{du}{2(x+1)} + \int \frac{1}{\tan^2 \theta + 1} \sec^2 \theta \, d\theta$$
$$= \int \frac{5}{u} du + \int 1 \, d\theta$$
$$= 5 \ln |u| + \theta + C$$
$$= 5 \ln (x^2 + 2x + 2) + \tan^{-1}(x+1) + C$$

Finally, our integral works out as

$$\int \frac{x^2 + x + 11}{(x+1)(x^2 + 2x + 2)} \, dx = 11 \ln|x+1| - 5\ln(x^2 + 2x + 2) - \tan^{-1}(x+1) + C$$

6.7 Improper Integrals. Sometimes, integrals that appear to be infinite in extent can be given a finite value in a way that seems sensible.

For example, consider

$$\int_{1}^{\infty} \frac{1}{x^2} \, dx$$

If we draw a picture of what this might mean graphically, in the same way as we dit for integrals $\int_a^b f(x) dx$ where a and b are finite, we should be looking at the area of the region under the graph

$$y = \frac{1}{x^2} \qquad x \ge 1$$

— a region that stretches infinitely far into the distance. So it seems infinite and nothing more to be said.



But, before we conclude that it is infinite, suppose we imagine colouring in the area under that graph with paint, and we do it so that we apply the paint evenly so that we use a fixed amount per square inch. The amount of paint we would need should be infinite if the area is infinite.

We would never be done painting an infinite area, an so we could paint a wide but finite section and see how much paint we need.



In this picture, we show colouring for $1 \le x \le 3$, but we could replace 3 by any finite b > 1. The area covered up as far as b works out as

$$\int_{1}^{b} \frac{1}{x^{2}} dx = \left[-\frac{1}{x} \right]_{1}^{b} = -\frac{1}{b} - (-1) = 1 - \frac{1}{b}$$

and we see that, rather than a huge answer, we always get an answer < 1. In fact as $b \to \infty$, the answer approaches 1. It does not tend to ∞ . So if we have enough paint to paint 1 square unit of area, we will never completely run out although there will be very little paint left when b is large. Maybe there is a case for deciding that $\int_{1}^{\infty} 1/x^2 dx$ should have the value 1.

We make this our *definition*. We *define*

$$\int_1^\infty \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2} \, dx$$

(which is 1).

More generally, if y = f(x) is defined and continuous for $x \ge a$ we define the improper integral

$$\int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{1}^{b} f(x) \, dx$$

if this limit exists and is *finite*. On the other hand if the limit does not exist at all, or is infinite, we say that the improper integral $\int_a^{\infty} f(x) dx$ fails to converge. Example. Find $\int_2^{\infty} 1/x dx$.

In this kind of example (an improper integral) we start by using the right definition. This shows that we realise that there is an issue about the integral making sense, and that we know how the issue is dealt with. We get

$$\int_{2}^{\infty} 1/x \, dx = \lim_{b \to \infty} \int_{2}^{b} 1/x \, dx$$
$$= \lim_{b \to \infty} [\ln x]_{2}^{b}$$
$$= \lim_{b \to \infty} \ln b - \ln 2.$$

If we look at the graph of $y = \ln x$ we see that this limit is ∞ . Because it is not finite, we say that the improper integral

$$\int_{2}^{\infty} \frac{1}{x} \, dx$$

does not converge.

Other types of improper integral

(i) If y = f(x) is defined and continuous for $x \le b$, then we define the improper integral

$$\int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx$$

if this limit exists and is finite. If the limit does not exist, or is infinite, we say that the improper integral $\int_{-\infty}^{b} f(x) dx$ does not converge.

(ii) If y = f(x) is continuous on a finite interval $a < x \le b$, excluding the left end point a (where it might have an asymptote or other bad behaviour), hen we define the improper integral

$$\int_{a}^{b} f(x) \, dx = \lim_{c \to a^{+}} \int_{c}^{b} f(x) \, dx$$

if this limit exists and is finite. If the limit does not exist, or is infinite, we say that the improper integral does not converge.

Example. Consider $\int_0^1 1/x^2 dx$.

Using the definition

$$\int_{0}^{1} 1/x^{2} dx = \lim_{c \to 0^{+}} \int_{c}^{1} 1/x^{2} dx$$
$$= \lim_{c \to 0^{+}} \left[-\frac{1}{x} \right]_{c}^{1}$$
$$= \lim_{c \to 0^{+}} -1 + \frac{1}{c}$$

As this limit is infinite, the improper integral $\int_0^1 1/x^2 dx$ does not converge.

(iii) If y = f(x) is continuous on a finite interval $a \le x < b$, excluding the right end point b (where it might have an asymptote or other bad behaviour), hen we define the improper integral

$$\int_{a}^{b} f(x) dx = \lim_{c \to b^{-}} \int_{a}^{c} f(x) dx$$

if this limit exists and is finite. If the limit does not exist, or is infinite, we say that the improper integral does not converge.

(iv) If an integral is improper for more than one reason, or at a place not one of the endpoints, we divide it as a sum of improper integrals of the types we have considered above. If each one of the bits has a finite value, then the value of the whole is the sum. But if any one of the bits fails to converge then the whole is said not to converge.

Examples

- (a) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$ (It does not matter about using 0 as a stopping point. Any finite point would do, for example $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{4} \frac{1}{1+x^2} dx + \int_{4}^{\infty} \frac{1}{1+x^2} dx$)
- (b) $\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx$

Here the problem with the integral is at 0, where the integrand has an asymptote. In fact we already worked out that $\int_0^1 1/x^2 dx$ does not converge and so we know that $\int_{-1}^1 \frac{1}{x^2} dx$ also does not converge.

This is an example where an unthinking use of integration would produce a wrong answer -2. (You might be suspicious if the integral of a positive thing in the left to right direction turned out to be negative.)

(c) $\int_{-\infty}^{\infty} \frac{1}{x(1+x^2)} dx = \int_{-\infty}^{1} \frac{1}{x(1+x^2)} dx + \int_{-1}^{0} \frac{1}{x(1+x^2)} dx + \int_{0}^{1} \frac{1}{x(1+x^2)} dx + \int_{1}^{\infty} \frac{1}{x(1+x^2)} dx$ The original is improper because of the $-\infty$ limit, the asymptotes at x = 0 and the ∞ limit. Each of the 4 bits above has just one problem at one end.

A Appendix

In 2006–7, this material was in 1S1.

A.1 Remark. The aim here is to explain how to find antiderivatives (indefinite integrals). By the fundamental theorem, this also allows us to find definite integrals exactly.

A.2 Remark. (i) Substitution. If we integrate both sides of the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
 where $y = f(u), u = u(x), y = f(u(x))$

we get

$$\int \frac{dy}{dx} dx = \int \frac{dy}{du} \frac{du}{dx} dx \tag{1}$$

and this means that if we we have an integrand we can express like the right hand side as

$$\int g(u) \frac{du}{dx} \, dx$$

then we can work it out if we have an antiderivative for g(u), that is if we can find f(u) with f'(u) = g(u) it will match exactly what we had in (1). It comes down to being able to cancel the dx's and

$$\int g(u)\frac{du}{dx}\,dx = \int g(u)\,du$$

Example. Consider the problem of finding

$$\int 3x^2 \cos(x^3) \, dx$$

If we notice that $u = x^3$ has $\frac{du}{dx} = 3x^2$ a factor in the integrand, then we can write

$$\int 3x^2 \cos(x^3) \, dx = \int \frac{du}{dx} \cos u \, dx = \int \cos u \frac{du}{dx} \, dx = \int \cos u \, du$$

This is something we can do as it is $\sin u + C$. So the answer is $\sin u + C$ but we should write the answer in terms of x (because the problem was in terms of x) and so we get

$$\int 3x^2 \cos(x^3) \, dx = \sin u + C = \sin(x^3) + C$$

Normally we write this out in a way like the following and rely on the possibility of treating dx like a numerical quantity that can be canceled between numerator and denominator.

$$\int 3x^2 \cos(x^3) dx \qquad \text{Let } u = x^3$$
$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2 dx$$
$$\frac{du}{3x^2} = dx$$
$$\int 3x^2 \cos(x^3) dx = \int 3x^2 \cos u \frac{du}{3x^2}$$
$$= \int \cos u \, du$$
$$= \sin u + C$$
$$= \sin(x^3) + C$$

The reason the substitution $u = x^3$ works is because the derivative du/dx is a factor in the integrand. For a substitution to work, it is usually necessary for du/dx to be a factor in the integrand (essentially, maybe in a slightly hidden way). It is vital that we express all x's and dx's in terms of u and du.

(ii) Integration by parts If we integrate both sides of the product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

we get

$$\int \frac{d}{dx}(uv) \, dx = \int v \frac{du}{dx} \, dx + \int u \frac{dv}{dx} \, dx$$

or

$$uv = \int v \frac{du}{dx} \, dx + \int u \frac{dv}{dx} \, dx$$

This allows us a way of transforming integrals of the form of a product of one function times the derivative of another

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

into a different integral (where the differentiation has flipped from one factor to the other). The advantage of this comes if we know how to manage the new integral (or at least if it is simpler than the original). The integration by parts formula is usually written with the ds's canceled

$$\int u\,dv = uv - \int v\,du$$

Examples

(a)
$$\int x \ln x \, dx$$

Solution: The two most obvious ways to use integration by parts are

- u = x, $dv = \ln x \, dx$ (Problem with this is we can't find v)
- $u = \ln x, dv = x dx$

It turns out that the second is good.

$$\int x \ln x \, dx \qquad \text{Let} \quad u = \ln x \quad dv = x \, dx$$
$$du = \frac{1}{x} \, dx \quad v = \frac{x^2}{2}$$
$$\int x \ln x \, dx = \int u \, dv$$
$$= uv - \int v \, du$$
$$= (\ln x) \frac{x^2}{2} - \int \frac{x^2}{2} \frac{1}{x} \, dx$$
$$= (x^2/2) \ln x - \int \frac{x}{2} \, dx$$
$$= (x^2/2) \ln x - \frac{x^2}{4} + C$$

(b) $\int_{1}^{e} \ln x \, dx$

Solution: This is one of a very few cases which can be done by taking dv = dx and u = the integrand. Every integral takes the form $\int u \, dv$ in that way, but it is rarely a good way to start integration by parts.

Here we can make use of the definite integral form of the integration by parts formula

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$

which arises in the same way as the indefinite integral formula (take definite integrals of the product rule for differentiation).

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$$\int_{1}^{e} \ln x \, dx \qquad \text{Let } u = \ln x \quad dv = dx \\ du = \frac{1}{x} \, dx \quad v = x \\ \int_{1}^{e} \ln x \, dx = \int_{1}^{e} u \, dv \\ = [uv]_{1}^{e} - \int_{1}^{e} v \, du \\ = [(\ln x)x]_{1}^{e} - \int_{1}^{e} x \frac{1}{x} \, dx \\ = e \ln e - \ln 1 - \int_{1}^{e} 1 \, dx \\ = e - [x]_{1}^{e} \\ = e - (e - 1) = 1 \end{aligned}$$

(c)
$$\int x^2 \cos x \, dx$$

Solution:

$$\int x^{2} \cos x \, dx \qquad \text{Let } u = x^{2} \quad dv = \cos x \, dx$$
$$du = 2x \, dx \quad v = \sin x$$
$$\int x^{2} \cos x \, dx = \int u \, dv$$
$$= uv - \int v \, du$$
$$= x^{2} \sin x - \int \sin x (2x) \, dx$$
$$= x^{2} \sin x - \int 2x \sin x \, dx$$

The point here is that we have succeeded in simplifying the problem. We started with x^2 times a trigonometric function $(\cos x)$ and we have now got to x times a trigonometric function $(\sin x \text{ this time, but that is not so different in difficulty to <math>\cos x$). If we continue in the *same* (or similar) way and apply integration by parts again, we can make the problem even simpler. We use U and V this time in case we might get confused with

the earlier u and v.²

$$\int 2x \sin x \, dx \qquad \text{Let } U = 2x \quad dV = \sin x \, dx$$
$$dU = 2 \, dx \quad V = -\cos x$$
$$\int 2x \sin x \, dx = \int U \, dV$$
$$= UV - \int V \, dU$$
$$= 2x(-\cos x) - \int (-\cos x) 2 \, dx$$
$$= -2x \cos x + \int 2 \cos x \, dx$$
$$= -2x \cos x + 2 \sin x + C$$

Combining with the first stage of the calculation

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x - C$$

and, in fact -C is plus another constant. Since C can be any constant, the answer

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

is also good.

We will not in fact learn any other techniques than these which are purely integration methods. We will spend some time explaining how to make use of these techniques in specific circumstances (as it is often not at all obvious how to do so). There is one other method we will come to called *partial fractions*, a method for integrating fractions such as

$$\int \frac{x+2}{(x-1)(x^2+2x+2)} \, dx$$

However, the thing we have to learn about is algebra — a way to rewrite fractions like this as sums of simpler ones — and there is no new idea that is directly integration. The algebra allows us to tackle problems of this sort.

A.3 Trigonometric Integrals. (i) Powers of $\sin x$ times powers of $\sin x$ with one power odd

Method: For

$$\int \sin^n x \cos^m x \, dx$$

• if n = the power of $\sin x$ is odd, substitute $u = \cos x$

²One thing to avoid is U = v and V = u because this will just unravel what we did to begin with.

• if m = the power of $\cos x$ is odd, substitute $u = \sin x$

Example: $\int \sin^3 x \cos^4 x \, dx$ Solution: Let $u = \cos x$, $du = -\sin x \, dx$, $dx = \frac{du}{-\sin x}$ $\int \sin^3 x \cos^4 x \, dx = \int \sin^3 x u^4 \frac{du}{-\sin x}$ $= \int -\sin^2 x u^4 \, du$ $= \int -(1 - \cos^2 x) u^4 \, du$ $= \int -(1 - u^2) u^4 \, du$ $= \int u^6 - u^4 \, du$ $= \frac{1}{7} u^7 - \frac{1}{5} u^5 + C$ $= \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C$

(ii) Powers of $\sin x$ times powers of $\sin x$ with both powers even Method: use the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Example:
$$\int \sin^4 x \cos^2 x \, dx$$

Solution:
 $\int \sin^4 x \cos^2 x \, dx = \int (\sin^2 x)^2 \cos^2 x \, dx$
 $= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \left(\frac{1}{2}(1+\cos 2x)\right) \, dx$
 $= \frac{1}{8} \int (1-2\cos 2x + \cos^2 2x)(1+\cos 2x) \, dx$
 $= \frac{1}{8} \int (1-\cos 2x - \cos^2 2x + \cos^3 2x \, dx)$

Now $\int 1 dx$ is no bother. $\int \cos 2x dx$ is not much harder than $\int \cos x dx = \sin x + C$. If we look at

$$\frac{d}{dx}\sin 2x = (\cos 2x)2$$

we can see that $\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$. (This can also be done by a substitution u = 2x but that is hardly needed.) Next

$$\int \cos^2 2x \, dx = \int \frac{1}{2} (1 + \cos 4x) \, dx = \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) + C$$

(using the same ideas as for $\int \cos 2x \, dx$).

For $\int \cos^3 2x \, dx$ we are in a situation where we have an odd power of \cos times a zeroth power of \sin . So we can use the earlier method (the fact that the angle is 2x doe snot make a big difference) of substituting $u = \sin 2x$. Then $du = 2\cos 2x \, dx$, $dx = \frac{du}{2\cos 2x}$,

$$\int \cos^3 2x \, dx = \int \cos^3 2x \, \frac{du}{2 \cos 2x}$$
$$= \frac{1}{2} \int \cos^2 2x \, du$$
$$= \frac{1}{2} \int 1 - \sin^2 2x \, dx$$
$$= \frac{1}{2} \int 1 - u^2 \, du$$
$$= \frac{1}{2} \left(u - \frac{1}{3} u^3 \right) + C$$
$$= \frac{1}{2} \sin 2x - \frac{1}{6} \sin^3 2x + C$$

Putting all the bits together

$$\int \sin^4 x \cos^2 x \, dx = \frac{1}{8} \int 1 - \cos 2x - \cos^2 2x + \cos^3 2x \, dx$$
$$= \frac{1}{8} \left(x - \frac{1}{2} \sin 2x - \frac{1}{2}x - \frac{1}{8} \sin 4x + \frac{1}{2} \sin 2x - \frac{1}{6} \sin^3 2x \right) + C$$
$$= \frac{1}{16} x - \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C$$

(iii) **Powers of** $\sin x$ and $\cos x$

Method: Use the previous two methods treating

$$\int \sin^n x \, dx = \int \sin^n x (\cos x)^0 \, dx$$

and similarly for $\int \cos^m x \, dx$ (that is treat the second power as the zeroth power). **Example**:

• $\int \cos^3 x \, dx$ Solution: $\int \cos^3 x \, dx = \int (\sin x)^0 \cos^3 x \, dx$. Power of \cos odd. Substitute $u = \sin x$, $du = \cos x \, dx$, $dx = \frac{du}{\cos x}$

$$\int \cos^3 x \, dx = \int \cos^3 x \, \frac{du}{\cos x}$$
$$= \int \cos^2 x \, du$$
$$= \int 1 - u^2 \, du$$
$$= u - \frac{1}{3}u^3 + C$$
$$= \sin x = \frac{1}{3}\sin^3 x + C$$

• $\int \sin^4 x \, dx$

Solution: Use even powers method.

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

= $\int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \, dx$
= $\frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x \, dx$
Note: still have one even power
= $\frac{1}{4} \int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x) \, dx$
= $\frac{1}{4} \int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \, dx$
= $\frac{1}{4} \left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$
= $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

There is another way to do these problems, and it is given in the book by Anton. Possibly also some of you know it already.

Alternative method: Use the reduction formulae

$$\int \sin^{n} x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$
$$\int \cos^{m} x \, dx = \frac{1}{m} \cos^{m-1} x \sin x + \frac{m-1}{m} \int \cos^{m-2} x \, dx$$

(valid for $n \ge 2$ and $m \ge 2$) to express the integrals of a power in terms of integrals where the power is reduced by 2. Using the formula enough times, will get down to power 1 or zero eventually.

Example:

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$
(using the case $n = 4$)
$$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(-\frac{1}{2} \sin x \cos x + \frac{1}{2} \int (\sin x)^0 \, dx \right)$$
(using the case $n = 2$)
$$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x - \frac{3}{8} \int 1 \, dx$$

$$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x - \frac{3}{8} x + C$$

Note that the answer looks different to the answer we got by the other method. They can be reconciled using trigonometry:

$$\sin 2x = 2 \sin x \cos x$$

$$\sin 4x = 2 \sin 2x \cos 2x$$

$$= 2(2 \sin x \cos x)(1 - 2 \sin^2 x)$$

$$= 4 \sin x \cos x - 8 \sin^3 x \cos x$$

$$\frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \frac{3}{8}x - \frac{1}{2} \sin x \cos x + \frac{1}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x$$

$$= \frac{3}{8}x - \frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x$$

The way to establish the reduction formulae is to use integration by parts. For example

$$\int \sin^{n} x \, dx = \int \sin^{n-1} x \sin x \, dx$$

$$= \int u \, dv$$

where $u = \sin^{n-1} x$ $dv = \sin x \, dx$
 $du = (n-1) \sin^{n-2} x \cos x \, dx$ $v = -\cos x$

$$= uv - \int v \, du$$

$$= \sin^{n-1} x (-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^{2} x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \cos^{2} x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^{n} x \, dx$$

Though this looks bad, we have in fact found an equation that is satisfied by $\int \sin^n x \, dx$ and if we solve the equation for this quantity in terms of the other things, we get

$$n\int \sin^{n} x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$
$$\int \sin^{n} x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

and that is the reduction formula as claimed. The one for powers of $\cos x$ works out in a similar way.