# Chapter 4: Linear approximation and applications

These are just summaries of the lecture notes, and not all details are included. Most of what we include here is to be found in more detail in Anton (that is Anton, Bivens and Davis).

**Remark 4.1** The linear approximation formula arises from the definition of the derivative of a function $y = f(x)$ at a point $x = a$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

It follows that for $x$ near $a$ (but $x \neq a$),

$$\frac{f(x) - f(a)}{x - a} \cong f'(a)$$

Multiply across by $x - a$ and rearrange to get

$$f(x) \cong f(a) + f'(a)(x - a) \qquad x \text{ near } a$$

This is the *linear approximation formula*.

$y = f(a) + f'(a)(x - a)$ is the equation of a line with slope $f'(x)$ and $(x, y) = (a, f(a))$ is one point on the line. It is the equation of the tangent line to the graph $y = f(x)$ at the point where $x = a$.

Graphically, the linear approximation formula says that the graph $y = f(x)$ is close to the graph of its tangent line $y = f(a) + f'(a)(x - a)$ if we stay near to the point of tangency $(x, y) = (a, f(a))$.

**Examples 4.2**   (i)  Find the linear approximation to $y = \sin x$ centered at $x = 0$.

Solution: this means to work out $f(x) \cong f(a) + f'(a)(x - a)$ with $f(x) = \sin x$ and $a = 0$. Result comes out as $\sin x \cong x$ (for small $x$).

(ii)  Find $\sqrt{9.02}$ approximately using linear approximation.

Solution: This means we use $f(x) = \sqrt{x}$ and $a = 9$ in the linear approximation formula. We end up with

$$\sqrt{x} \cong \sqrt{9} + \frac{1}{2\sqrt{9}}(x - 9) = 3 + \frac{1}{6}(x - 9)$$

for $x$ near 9 and so $\sqrt{9.02} \cong 3 + \frac{1}{6}(0.02) = 3.003\dot{3}$. (Calculator give $\sqrt{9.02} = 3.003331484$.)

Note: This type of example is not very important, but they are easy exercises to get used to the formula.

**Remark 4.3** *Why is the linear approximation formula good?*

We saw that $\sin x \cong x$ for $x$ small, but presumably $\sin x$ is also not so far from $2x$ when $x$ is small. What makes $\sin x \cong x$ superior?

To answer this we go back to the derivation of the linear approximation formula and redo it so as to avoid making any approximation until the end.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

It follows that for $x$ near $a$ (but $x \neq a$),

$$\frac{f(x) - f(a)}{x - a} = f'(a) + E(x)$$

(where $E(x)$ is just the difference between the the other two things). We know that $E(x)$ is small if $x$ is close to $a$ (and $x \neq a$).

Multiply across by $x - a$ and rearrange to get

$$f(x) = f(a) + f'(a)(x - a) + E(x)(x - a)$$

In the *linear approximation formula*, we discard $E(x)(x - a)$ and this is small for two reasons. For $x$ near $a$, $E(x)$ is small and so is $x - a$. We are throwing away something that is small **compared to the already small** $x - a$ when we say

$$f(x) \cong f(a) + f'(a)(x - a) \qquad (x \text{ near } a).$$

No other line has the properties that

- exactly matches $y = f(x)$ at $x = a$;

- difference small compared to $x - a$ for $x$ near $a$.

Another way to explain what is good about linear approximation is to use a more sophisticated fact, Taylor's theorem or Taylor's formula for the remainder. (This has been mentioned in 1S1 already.) It says that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c)}{2!}(x - a)^2$$

for some $c$ between $a$ and $x$. This needs $f''$ to make sense, but if we know that $f''(c)$ cannot be too large, no matter where $c$ is in the range between $a$ and $x$, this tells us that $E(x)(x - a) = \frac{f''(c)}{2!}(x-a)^2$ is at its biggest proportional to $(x-a)^2$. If $x-a$ is small, then $(x-a)^2$ is even smaller. For example if $x-a = 0.1$, then $(x-a)^2 = 0.01$. If $x-a = 0.01$ then $(x-a)^2 = 0.0001 = 10^{-4}$.

**Remark 4.4** *Importance of the linear approximation.*

The real significance of the linear approximation is the use of it to convert intractable (nonlinear) problems into linear ones (and linear problems are generally easy to solve).

For example the differential equation for the oscillation of a simple pendulum works out as

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$$

where $\theta = \theta(t)$ is the angle of the pendulum from the vertical at time $t$. $g$ is the acceleration due to gravity (about $9.81m/s^2$ at sea level) and $\ell$ is the length of the pendulum.

This is a nonlinear equation and solutions cannot be written down in any simple way.

If we assume that $\theta(t)$ is always small (pendulum does not swing too wildly) then we can say $\sin \theta \cong \theta$ and the differential equation is approximately

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

This equation is linear and is relatively easy to solve. All the solutions are of the form

$$\theta(t) = C_1 \cos kt + C_2 \sin kt$$

where $C_1$ and $C_2$ can be any constants and $k = \sqrt{g/\ell}$.

Of course this is only approximately right, but the approximation turns out to be very accurate in this case.

**Relative Errors 4.5** If we measure (or compute) a quantity where the 'true' or 'correct' answer is $x$ but we get a slightly different answer $\tilde{x}$ (maybe because of inaccuracies in an experiment or because we made some rounding errors in the calculation) then the *error* is the difference

$$\text{error} = x - \tilde{x} = (\text{true value}) - (\text{approximate value})$$

Normally we don't worry about the sign and only concentrate on the magnitude or absolute value of the error. In order to asses the significance of the error, we *have to compare it to the size of the quantity* $x$.

The *relative error* is a more significant thing:

$$\text{relative error} = \frac{\text{error}}{\text{true value}} = \frac{(\text{true value}) - (\text{approximate value})}{\text{true value}} = \frac{x - \tilde{x}}{x}$$

It expresses the error as a fraction of the size of the thing we are aiming at. 100 times this give the percentage error.

**Example 4.6** Suppose we use $\frac{22}{7}$ as an approximation to $\pi$. Then the relative error is

$$\text{relative error} = \frac{(\text{true value}) - (\text{approximate value})}{\text{true value}} = \frac{\pi - \frac{22}{7}}{\pi} = 0.000402$$

or $0.04\%$.

**Condition Numbers 4.7** We can use linear approximation to understand the following problem.

Say we measured $x$ but our answer was $\tilde{x}$ and then we compute with that to try to find $f(x)$ (some formula we use on our measurement). If there are no further approximation in the calculation we will end up with $f(\tilde{x})$ instead of $f(x)$. How good an approximation is $f(\tilde{x})$ to the correct value $f(x)$?

We assume that $\tilde{x}$ is close to $x$ and so linear approximation should be valid. We use the linear approximation formula

$$f(x) \cong f(a) + f'(a)(x - a) \qquad x \text{ near } a$$

with $x$ replaced by $\tilde{x}$

$$f(\tilde{x}) \cong f(a) + f'(a)(\tilde{x} - a) \qquad \tilde{x} \text{ near } a$$

and then $a$ replaced by $x$

$$f(\tilde{x}) \cong f(x) + f'(x)(\tilde{x} - x)$$

So the final error

$$(\text{true value}) - (\text{approximate value}) = f(x) - f(\tilde{x}) \cong f'(x)(x - \tilde{x})$$

Notice that $x - \tilde{x}$ is the error in the initial measurement and so we see that the derivative $f'(x)$ is a magnifying factor for the error.

*But* we are saying above that relative errors are more significant things than actual errors. So we recast in terms of relative errors. The relative error in the end (for $f(x)$) is

$$\frac{f(x) - f(\tilde{x})}{f(x)} \cong \frac{f'(x)}{f(x)}(x - \tilde{x})$$

To be completely logical, we should work with the relative error at the start $\frac{x - \tilde{x}}{x}$ instead of the actual error $x - \tilde{x}$. We get

$$\frac{f(x) - f(\tilde{x})}{f(x)} \cong \frac{x f'(x)}{f(x)} \frac{x - \tilde{x}}{x}$$

or

$$\text{relative error in value for } f(x) = \frac{x f'(x)}{f(x)} (\text{relative error in value for } x)$$

Thus the relative error will be magnified or multiplied by the factor $\frac{x f'(x)}{f(x)}$ and this factor is called the *condition number*.

In summary

$$\text{condition number} = \frac{x f'(x)}{f(x)}$$

**Examples 4.8**    (i)  Find the condition number for $f(x) = 4x^5$ at $x = 7$.

$$\frac{x f'(x)}{f(x)} = \frac{x(20x^4)}{4x^5} = \frac{20x^5}{4x^5} = 5.$$

So in this case it does happens not to depend on $x$.

(ii) Use the condition number to estimate the error in $1/x$ if we know $x = 4.12 \pm 0.05$.

If we take $f(x) = 1/x$ we can work out its condition number at $x = 4.12$:

$$\frac{xf'(x)}{f(x)} = \frac{x\left(\frac{-1}{x^2}\right)}{\frac{1}{x}} = \frac{\frac{-1}{x}}{\frac{1}{x}} = -1$$

This means it does not depend on 4.12 in fact.

Now our initial value $x = 4.12 \pm 0.05$ means we have a relative error of (at most)

$$\frac{\pm 0.05}{4.12} \cong \pm 0.012$$

The relative error in $f(4.12) = 1/(4.12)$ is then going to be about the same (because the condition number is $-1$ and this multiples the original relative error). So we have, using $\tilde{x} = 4.12$ as our best approximation to the real $x$,

$$f(\tilde{x}) = \frac{1}{\tilde{x}} = \frac{1}{4.12} = 0.242718$$

and this should have a relative error of about 0.012. The magnitude of the error is therefore (at worst) about $(0.012)(0.242718) = 0.0029$ or about 0.003. So we have the true value of

$$\frac{1}{x} = 0.242718 \pm 0.003 \text{ or, more realistically } 0.0243 \pm 0.003$$

(no point in giving the 718 as they are not at all significant).

(iii) $f(x) = e^x$. Condition numbers at $x = 10/3$ is $xf'(x)/f(x) = xe^x/e^x = x = 10/3 = 3.333\dot{3}$. So if we use $\tilde{x} = 3.33$ instead of $x = 10/3$ we would have a relative error to begin with

$$\text{relative error in } x = \frac{\frac{10}{3} - 3.33}{\frac{10}{3}} = 0.001$$

(that is an error of 0.1%). If we now compute $e^{3.33}$ while we ought to have computed $e^{10/3}$ we will have a relative error about 10/3 times larger (the condition number is 10/3, or roughly 3). So we will end up with a 0.3% error. In other words, still quite small.

If instead we were working with $x = 100/3$ and we took $\tilde{x} = 33.3$, we would have the same initial relative error

$$\text{relative error in } x = \frac{\frac{100}{3} - 33.3}{\frac{100}{3}} = 0.001$$

but the condition number for $e^x$ is now $x \cong 33$. The error in using $e^{33.3}$ where we should have had $e^{100/3}$ will be a relative error about 33 times bigger than the initial one, or $33 \times 0.001 = 0.033$. This means a 3.3% error (not very large perhaps, but a lot larger than the very tiny 0.1% we began with.

In fact $e^{33.3} = 2.89739 \times 10^{14}$ and 3.3% of that is $9.56137 \times 10^{12}$.

(iv) For $f(x) = x^2 - 10x + 11$, find the condition number at $x = 10$.

Answer: $\dfrac{xf'(x)}{f(x)} \mid_{x=10} = \dfrac{100}{11} \cong 9$.

**Root finding methods 4.9** *We look into some methods for finding solutions of equations, something that is often required. We will look at two fairly basic methods, the bisection method and Newton's method.*

These are approximate methods, capable of finding solutions to equations $f(x) = 0$ to any desired number of decimal places, but they never give and exact answer (except by a fluke).

Unlike quadratic equations $ax^2 + bx + c = 0$ (with $a \neq 0$ to make it genuinely quadratic), where we have a formula for the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we rarely are in a position to find an exact formula for a solution of an equation $f(x) = 0$ (if $f(x)$ is at all complicated, and real practical problems tend to be complicated).

For example, consider

$$x = \cos x$$

There is no formula for the solutions of this. To make it fit our methods we should write it as

$$x - \cos x = 0$$

so that it is $f(x) = 0$ with $f(x) = x - \cos x$.

A question that might arise is this: How can we know there are solutions of an equation where we don't have a formula? Sometimes there will not be any. For example, even in the case of quadratic equations, there might be none if we insist on looking for real number solutions and $b^2 - 4ac < 0$. (There are complex number solutions in this case.)

With $x - \cos x = 0$ we can convince ourselves graphically that there are solutions. That is, we can use the Intermediate Value Theorem (which we saw earlier in Chapter 3, but here it is again).

**Theorem 4.10 (Intermediate Value Theorem)** *If $y = f(x)$ is continuous for all $x$ in the finite closed interval $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs (a short way to write that is $f(a)f(b) < 0$) then there must be some $c \in (a, b)$ with $f(c) = 0$.*

**Bisection Method 4.11** The *bisection method* is a basic method, which is less labour intensive than accurately plotting the graph (and reading off the $x$ values where the graph crosses $y = 0$ as you did in school).

It is an approximation method, which will never get the exact answer (except by accident) but it can get the answer to any desired accuracy (or any number of decimal places) if used for long enough.

It is a method for locating roots of an equation $f(x) = 0$. To start it, we need a range $a \leq x \leq b$ where $f(x)$ is continuous and where $f(x)$ changes sign. So assume $f(a)f(b) < 0$ (meaning that the signs of $y = f(a)$ and $y = f(b)$ are opposite.

**Basic step:** Look at the mid point $x = c = \dfrac{a+b}{2}$. If $f(a)f(c) < 0$ then there is a root in the interval $(a, c)$. If not, then $f(c)f(b) \leq 0$ and there is a root in $[c, b)$.

Having a root exactly at $x = c = (a+b)/2$ is essentially impossible (it could happen by a fluke).

So the result of this basic step is to find that $y = f(x)$ has a root in either one of the two intervals $[a, c]$ or $[c, b]$.

**Method.** Replace the original interval $[a, b]$ by the half where there is a sign change and repeat the basic step.

Keep repeating until the root is located within a sufficiently small tolerance.

**Example 4.12** *For the equation* $\cos x = x$, *use the bisection method to locate a root.*
*Solution:* We have to have an equation $f(x) = 0$ and so take $f(x) = \cos x - x$.

We can start with $[a, b] = [0, 1]$ because $f(0) = 1 > 0$ while $f(1) = \cos 1 - 1 = -0.459698 < 0$ (remember radians). $f(x)$ is continuous on $[0, 1]$ (in fact everywhere).

Using the method 10 times we would get the following intervals

$$[0.5, 1], \qquad [0.5, 0.75], \qquad [0.625, 0.75], \qquad [0.6875, 0.75],$$
$$[0.71875, 0.75], \qquad [0.734375, 0.75], \qquad [0.734375, 0.742188], \qquad [0.738281, 0.742188],$$
$$[0.738281, 0.740234], \qquad [0.738281, 0.739258]$$

(Each time the interval shortens by half. We could continue to get it even shorter.)

Here is the graph in this case, so you can see what is going on.
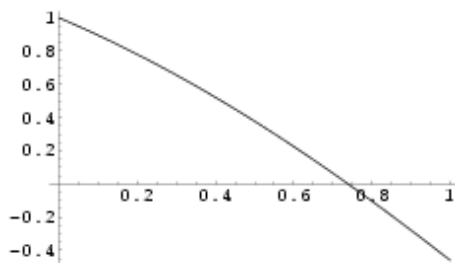


**Newtons Method 4.13** This is another method aimed at finding solutions to equations $f(x) = 0$ approximately. It has a more sophisticated idea behind it and generally works very well. However, it is not guaranteed to work (as the bisection method is). Newtons method uses linear approximation (and so requires $f'(x)$ to exist).

To begin with we assume we have a rough idea where there is a solution. We call this our initial guess and write $x = x_0$ for this value of $x$. (The effect of a bad or wild guess is often not serious. Typically a better guess will save effort and get to the answer more quickly, but there are cases where a bad guess will just not work. In those cases, you discover that the guess is bad as you go along. There are some cases where the method won't work, even with a pretty good guess.)

*The idea behind the method is as follows.* We have no formula for a solution of the correct equation $f(x) = 0$. (If we had a formula, we could just use it.) But linear equations are easy

to solve. So we replace the correct equation $f(x) = 0$ by its linear approximation centered at $x = x_0$. That is we solve

$$
\begin{aligned}
f(x_0) + f'(x_0)(x - x_0) &= 0 \\
f'(x_0)(x - x_0) &= -f(x_0) \\
x - x_0 &= -\frac{f(x_0)}{f'(x_0)} \\
x &= x_0 - \frac{f(x_0)}{f'(x_0)}
\end{aligned}
$$

This is not likely to be the correct solution of the real equation $f(x) = 0$ but the idea is that it should be a 'better guess' than $x = x_0$ was. We put

$$
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
$$

as an improved guess.

Then we repeat the idea starting with $x = x_1$ where we had $x = x_0$ and we get a further 'improved' guess

$$
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}
$$

Then we go again, and again. We get a sequence of improved guesses $x = x_1$, $x = x_2$, ..., where each new guess $x_{n+1}$ is related to the previous one by

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \ldots) \tag{1}
$$

Here is a graphical view of what happens. $f(x) = \cos x - 3x^2$, $x_0 = 0.2$, $x_1 = 0.814918$ and $x_2 = 0.582355$ in this example.
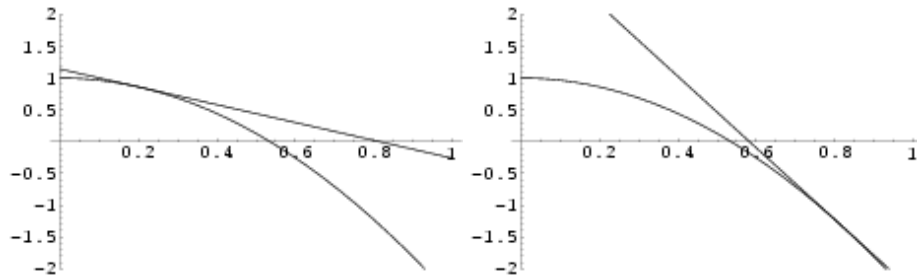


If we ever reached a true solution $f(x_n) = 0$ and then $x_{n+1} = x_n$. This is most unlikely, but typically we end up with $x_{n+1} \cong x_n$ for a fairly small $n$. $x_{n+1} \cong x_n$ then means

$$
\frac{f(x_n)}{f'(x_n)} \cong 0 \Rightarrow f(x_n) \cong 0
$$

So the method is

(i) Start with a guess $x = x_0$ of the solution of $f(x) = 0$

(ii) Find $x_1, x_2, \ldots$ using (1) until the values stabilise. When they stabilise we are pretty much at a solution.

(If they don't stabilise, try starting again with another guess.)

**Example 4.14** *For the equation* $\cos x = x$*, use Newtons method to locate a root.*
*Solution:* We have to have an equation $f(x) = 0$ and so take $f(x) = \cos x - x$.

We know (or can rediscover easily) from the Intermediate Value Theorem that there is a root in the range $0 < x < 1$. A reasonable guess then is the mid point $x_0 = 0.5$. (We could do better because of our earlier work with the example, but let's forget that work for now.)

Here are the values that result

| $n$ | $x_n$ |
|---|---|
| 1 | 0.755222 |
| 2 | 0.739142 |
| 3 | 0.739085 |
| 4 | 0.739085 |

We see that in a very few steps we have an answer that seems accurate to 5 places of decimals. We can calculate $f(x_4) = f(0.739085) = 2.2295 \times 10^{-7}$. So we seem to be pretty close.

*Mathematica* can work this out for us if we want it to.

```
f[x_]  =  Cos[x]  -  x
g[x_]  =  x  -  f[x]/f'[x]
x0  =  0.5
Do[Print[i, " , ", x0= g[x0]], {i, 1, 5}]
```

Here we just gave it the formulae from the method and then the `Do` command tells it to work out the new values of $x$ (or new guess) 5 times. The `Print` command is so we can get it to tell us the values as it goes along. The idea of `x0 = g[x0]` is that we keep calling the new value `x0`. This is easier to program and saves computer memory because we don't need the old values again. Here is the output:

```
1 ,  0.755222
2 ,  0.739142
3 ,  0.739085
4 ,  0.739085
5 ,  0.739085
```