Chapter 3. Graphing, maxima and minima

Most of this material is covered in Anton (Anton, Bivens & Davis).

These notes are not complete and skip some of the explanations from the lectures.

We begin with some important theorems about continuous functions. In one way they seem almost obvious, but they are not really obvious. An explanation for this statement is that the proofs of the theorems are quite complex (and beyond this course).

A theorem is a mathematical version of a law of nature, like Newtons Laws or Boyle's law. To say the theorem is true means that it is a rule (like a law of nature) that is always obeyed, a conclusion that is always true provided we are in the circumstances given. A proof is a reason why the theorem as stated has to be true. In mathematics we reason our way from the hypotheses to the conclusion. By contrast, in science laws are generally formulated as a result of many observations and then validated by comparing them with experimental results.

We start by recalling the definition of a continuous function, which you should recall from course 1S1.

3.1 Definition. Suppose that $S \subset \mathbb{R}$ is a set and $f \colon S \to \mathbb{R}$ is a function with domain S. We say that f is *continuous at a point* $x_0 \in S$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

We say f is *continuous* (on S) if it is continuous at **each** point $x_0 \in S$.

If we want to be fussy, we should take care of end points of S in a different way.

Interpreting the definition literally, continuity at the point x_0 says that if we change x from x_0 by a little, then $f(x)$ will only change a little from $f(x_0)$. It would seem normal that a tiny change in x should make only a small difference to the value $f(x)$, but there are examples where that is not the case.

Generally we explain the meaning of the term continuous like this. We say that a function is continuous if you can draw its graph without lifting your pen. In fact this is a somewhat incomplete way of saying what a continuous function is, but it at least fits with the word. It also fits with the fact that there are examples of functions that are not continuous because they have jumps in them.

Here is one such example.

3.2 Example. Say we want to define the sign of a number x as a function. We might define it like this. The function $\operatorname{sgn} \colon \mathbb{R} \to \mathbb{R}$ is given by the rule

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In one way the example might seem odd, but if you think about it a little it makes reasonable sense. Certainly the sign of a positive number should be $+$ and so having the value $+1 = 1$ seems right. And for x negative it is also reasonable to have the sign be -1. As for the sign of 0, we could maybe argue that it has no sign, or we could include it as positive. What we have done here is use a compromise value 0 (which we could interpret as no sign). In fact, no matter what we do, if we give any meaning to $\mathrm{sgn}(0)$, the example will still be not continuous at 0.

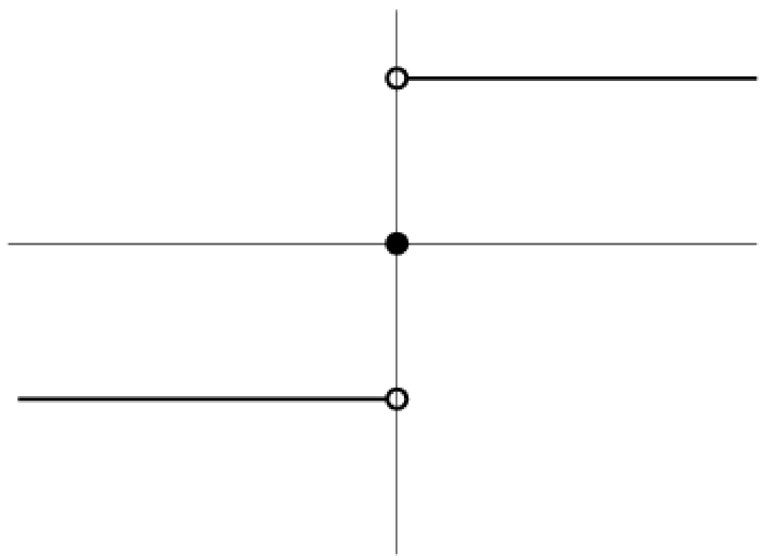
The reason is that

$$\lim_{x\to 0} \mathrm{sgn}(x)$$

does **not** exist because

$$\lim_{x\to 0^+} \mathrm{sgn}(x) = 1, \qquad \lim_{x\to 0^-} \mathrm{sgn}(x) = -1.$$

We know that if the (ordinary two-sided) limit as $x \to 0$ did exist then the two one-sided limits would have to be the same. As they are different there is no (two-sided) limit.



You can see from the graph that changing x ever so slightly from $x = 0$ will change the value of $\mathrm{sgn}(x)$ abruptly (from 0 to 1 or -1).

This function is not continuous at 0 (and so not continuous overall).

3.3 Examples. Despite the example we just saw, most functions you can write down by a formula are continuous (as long as we avoid dividing by zero). So polynomial functions like

$$\begin{aligned} f(x) &= 3x^2 - 11x + 12 \\ g(x) &= -31x^3 + x^2 + 5x + 77 \\ h(x) &= = x^4 - x^2 + x - 3 \end{aligned}$$

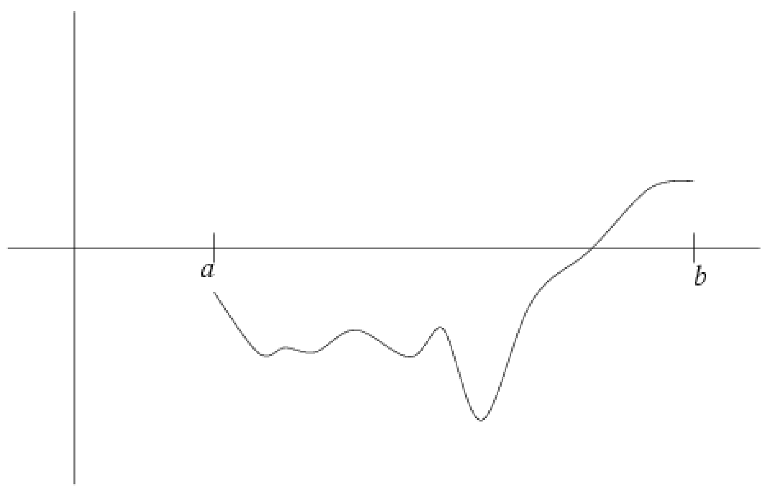
are continuous. We know enough about limits to know that $\lim_{x\to x_0} f(x) = f(x_0)$ for every $x_0 \in \mathbb{R}$. That is we find the limit as $x \to x_0$ in this case by plugging in $x = x_0$.

For the other examples $g(x)$ and $h(x)$ we can say the same. So all are continuous (everywhere on their domain $\mathbb{R}$.)

3.4 Theorem (Intermediate Value Theorem)**.** *If $y = f(x)$ is continuous for all x in the finite closed interval $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs (a short way to write that is $f(a)f(b) < 0$) then there must be some $c \in (a, b)$ with $f(c) = 0$.*

3.5 Remark. An actual proof of that is beyond us. A proof would be an argument to show that in any example whatever of a continuous function where the assumption that $f(a)$ and $f(b)$ have opposite signs, we can be sure that the conclusion will hold.

However, we can try to explain what the theorem says, and this is always the first step in going about a proof. We have to consider a totally unknown function $y = f(x)$. We have to have two things true about it, first that it is continuous (*for every single x in the interval $a \leq x \leq b$*) and second that the signs of $f(a)$ and $f(b)$ are opposite. Try to think of a complicated picture (or graph). Here is one fairly random picture.

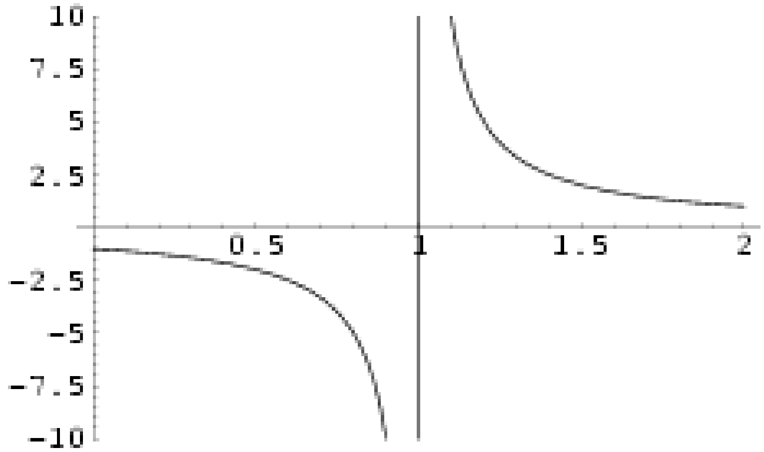


We started with $f(a) < 0$ and drew a graph ending up with $f(b) > 0$. So the graph started below the x-axis and ended up above it. The theorem just says that we have to cross the axis somewhere along the way.

Seems obvious?

Well, look back at the graph of the sgn function and shift it down by $1/2$. Take $f(x) = \operatorname{sgn}(x) - 1/2$ for $-1 \leq x \leq 1$. We would start at $y = -3/2$ and end at $y = 1/2$. But there is nowhere where $f(x) = 0$. Why not? Well we allowed a jump in the graph at $x = 0$, that is a discontinuity. Just one point where the graph is not continuous and the conclusion of the theorem does not hold.

Maybe a slight less artificial looking example is $f(x) = 1/(x - 1)$, $a = 0$ and $b = 2$. We have $f(a) = f(0) = -1 < 0$ and $f(b) = f(2) = 1 > 0$. But there is nowhere where we can say $f(x) = 0$ because $\frac{1}{x-1} \neq 0$. This $f(x)$ is continuous everywhere it makes sense. The only problem is that it fails to make sense at $x = 1$, which is one point in between a and b.



These examples might make you wonder if the theorem is really obvious. It **is** more or less obvious if you believe the pictorial interpretation that a continuous function is one where you can draw the graph without lifting your pen.

The picture has to be interpreted carefully because the graph $y = 1/(x-1)$ which we drew above is continuous and yet you need to draw the bit with $x < 1$ and the part with $x > 1$ separately. So you do have to lift your pen, but that is kind of explained because $x = 1$ is not included.

The definition of what a continuous function is, the one with limits, seems far from what we say about drawing the graph. The Intermediate Value Theorem is really a part of the justification for the thing about not lifting your pen.

We will come back to using the Intermediate Value Theorem in a reasonably practical way to guarantee that certain equations have solutions.

Our next big theorem is another justification for the assertion that continuous functions have graphs that can be drawn without lifting your pen (at least if we stick to intervals as the domain).

3.6 Max and min points. A point $x = c$ is an *absolute maximum point* for the function $y = f(x)$ if $f(c) \geq f(x)$ for every x in the domain of f.

A point $x = c$ is an *absolute minimum point* for the function $y = f(x)$ if $f(c) \leq f(x)$ for every x in the domain of f.

It is quite easy to understand the importance of finding these points. If we imagine that $f(x)$ is the cost of building something and x is a parameter that can be controlled, then it would be natural to want to minimise the cost by choosing the most efficient x. For other problems, you might want to choose x so as to get the most out if $f(x)$ represents output.

You can also imagine problems where there is more than one thing to vary. That would amount to a function of more than one variable and this is beyond us in this course.

3.7 Theorem. *If $f(x)$ is continuous for* every *point x in a **finite closed interval** $a \leq x \leq b$ (or we could just write, "for all $x \in [a,b]$" to put it more succinctly), then $f(x)$ has an absolute max point and an absolute min point on $[a,b]$.*

The theorem is quite difficult to prove, but it is quite useful. It says that certain max-min problems have a solution. The fact that there is a solution is a help because of the 'needle in the haystack' analogy. If you are searching for a needle in a haystack, you have a hard job. But if you are not sure whether there is one or not, you could be searching for ever and not know whether you might find the answer if you persevere or whether there is no answer.

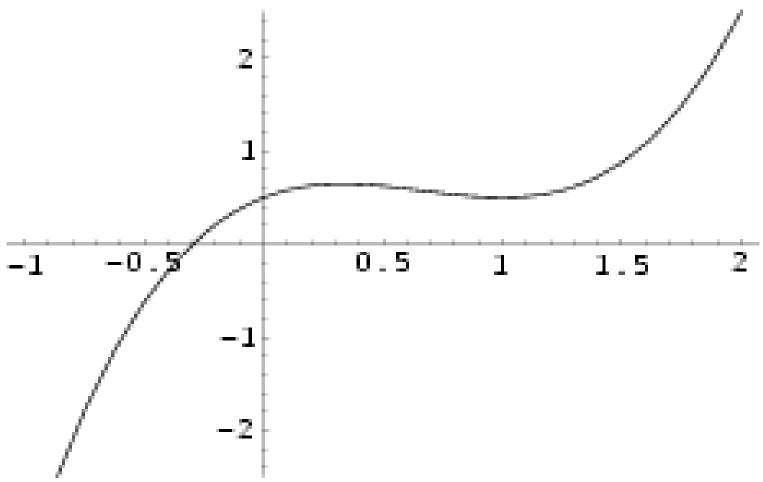
The method we will use for finding the max and min points will rely on calculus and also a concept that is weaker than absolute max (or min) points.

3.8 Definition. Let $y = f(x)$ be a function. Then a point $x = c$ is a *local max point* (or *relative max point*) for $f(x)$ if $f(c) \geq f(x)$ for all x nearby c

[Technically, we should say $f(c) \geq f(x)$ for all x within some positive distance of c on either side of c]

A point $x = c$ is called a *local min point* (or a *relative min point*) for $f(x)$ if $f(c) \leq f(x)$ for all x nearby c.

3.9 Examples. (i) Consider this example graph: $y = f(x) = x^3 - 2x^2 + x + 1/2$ for $-1 \leq x \leq 2$



Absolute max at $x = 2$ (an end point, not a local max by our definition because we insisted on $f(c) \geq f(x)$ for all x near c on either side of c).

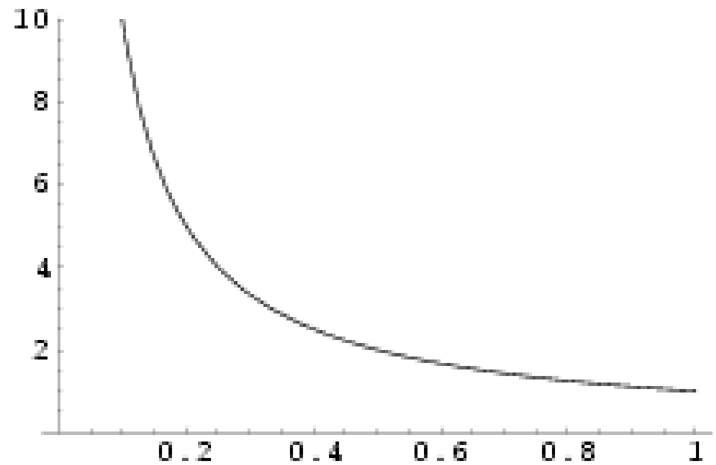
Local max at $x = 1/3$ (not so easy to see for sure where it is, but we will check this).

Absolute min $x = -1$, local min $x = 1$ (again not quite clear).

Notice that the theorem applies, but the absolute max and min points are at the endpoints

If we look at the sign of $dy/dx = f'(x) = 3x^2 - 4x + 1 = (3x - 1)(x - 1)$ we can work out that $f'(x) > 0$ for $x < 1/3$ and also for $x > 1$. In between, in $1/3 < x < 1$, $f'(x) < 0$. We conclude then that the graph is increasing for $x < 1/3$ (in fact even for $x \leq 1/3$), decreasing on the interval $1/3 \leq x \leq 1$ and increasing again for $x \geq 1$. This way we can justify the statements about the local min and local max points.

(ii) $f(x) = \frac{1}{x}$ $\qquad 0 < x < 1$



In this graph there is no max. The theorem does not apply because the *interval is not closed* (the endpoints are not included). The graph also has no minimum because the point $x = 1$ is not included (and you could argue that this is the result of a somewhat artificial restriction on x).

3.10 Definition. A *critical point* for a function $y = f(x)$ is a values of x where $f'(x) = 0$.

3.11 Theorem. *Suppose $y = f(x)$ has a local max (or min) at $x = c$. If f is differentiable at $x = c$ and $x = c$ is an interior point of the domain of f (that is not an end point) then $f'(c) = 0$.*

Proof. We will include a proof for the local max case. The local min case is quite similar.

We know from the definition that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

We also know that when we have a (two-sided) limit like this, then we also have one-sided limits that turn out to be the same limit $f'(c)$. So $f'(c)$ is equal to the right sided limit:

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

Assuming $x = c$ is a local max, we know that $f(c) \geq f(x)$ when x is nearby c. That means for h near 0 we have $f(c) \geq f(c+h)$ and so $f(c+h) - f(c) \leq 0$. When we take $h > 0$ small, we then get

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Taking the limit as $h \to 0^+$, we see

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

On the other hand $f'(c)$ is also the left sided limit

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We can still argue as before that $f(c+h) - f(c) \leq 0$ when h is small. However when $h < 0$ and small, dividing by h reveres the inequality and so we get

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

(for $h < 0$ and small). Taking the limit as $h \to 0^-$ we see

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

From one direction we concluded $f'(c) \leq 0$ and from the other $f'(c) \geq 0$. The only way to reconcile these two is to have $f'(c) = 0$.

[Notice that we needed to use both sides to get the conclusion. That is why we need to be at an interior point.]

The modifications needed to show that $f'(c) = 0$ at local min points that are interior points are as follows. We have $f(c) \leq f(c+h)$ for h small, and so $f(c+h) - f(c) \geq 0$. Thus

$$\frac{f(c+h) - f(c)}{h} \geq 0 \text{ for } h > 0 \text{ small}$$

and so $f'(c) = \lim_{h \to 0^+}(f(c+h) - f(c))/h \geq 0$. Working on the other side (as $h \to 0^-$) we get $f'(c) \leq 0$. Thus, again $f'(c) = 0$. □

3.12 Sign of the derivative. We know that if $y = f(x)$ is a function that has a derivative (we call this a *differentiable function*) then the value $f'(a)$ of the derivative at a point $x = a$ is the slope of the tangent line to the graph $y = f(x)$ at the point on the graph where $x = a$ (which is the point $(x, y) = (a, f(a))$).

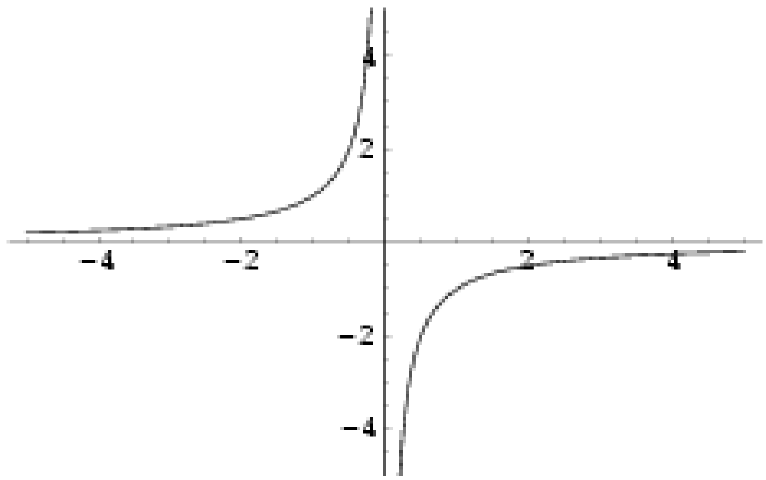
If $f'(a) > 0$ is positive, then the tangent line is sloping upwards there, but we cannot conclude much from knowing the sign of the derivative at one point. We can however conclude something if we know that $f'(x) > 0$ is an interval of values of x.

3.13 Definition. Let $f\colon S \to \mathbb{R}$ be a function defined on a domain $S \subset \mathbb{R}$. Then $f(x)$ is called *increasing* if whenever $x_1 < x_2$ (with $x_1, x_2 \in S$) then $f(x_1) < f(x_2)$.

A way to think about this is to say that if you move left to right on the graph $y = f(x)$, you always end up at a higher point than where you started.

3.14 Example. Consider the function $f(x) = -1/x$, a function which naturally makes sense for all $x \neq 0$. So we should take $S = \{x \in \mathbb{R} : x \neq 0\}$ (every x except $x = 0$) and define $f\colon S \to \mathbb{R}$ by the rule $f(x) = -1/x$.

We can compute $f'(x) = 1/x^2$ and we see that it is always positive. However, if we look at the graph, we can see that the function is *not increasing* overall. Look at $x_1 = -1$, $x_2 = 1$, where $x_1 < x_2$ but $f(x_1) = 1 > f(x_2) = -1$.



What we see in the next theorem is that this kind of example happens because of the gap at $x = 0$ (which allows the graph to jump down in this case).

3.15 Theorem. *If $y = f(x)$ is differentiable ($f'(x)$ exists) for* all x *in an* interval *and if $f'(x) > 0$ for all x in the interval, then $f(x)$ is* ***increasing*** *on that interval.*

Although this may seem reasonably obvious, the previous example shows that it is not so obvious. We need the assumption about intervals for it to be true.

It can be *proved* to be correct using a theorem called the Mean Value Theorem (for derivatives). We will come back to explain that soon. For now, we will consider the link in the other direction.

3.16 Proposition. *If f is increasing on an interval and $f'(x)$ exists, then $f'(x) \geq 0$.*

This Proposition is much easier to prove. It is not quite a converse of the Theorem because of the greater than or equal (where there was $f'(x) > 0$ in the theorem).

We will not give the relatively easy proof of the Proposition in full detail, but it is basically a consequence of the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

When $h > 0$, $x + h > x$ and so $f(x + h) > f(x)$ by the increasing property. So

$$\frac{f(x+h) - f(x)}{h} > 0.$$

When $h \to 0^+$, the limit of this could be 0 but it cannot be negative. So $f'(x) \geq 0$.

If we change the perspective slightly from increasing in the strict sense to *non-decreasing* functions, we can get exactly converse statements.

- For $f\colon S \to \mathbb{R}$ a function on $S \subset \mathbb{R}$, f if called *non-decreasing* is whenever $x_1, x_2 \in S$ and $x_1 < x_2$, then $f(x_1) \leq f(x_2)$

- If $f'(x) \geq 0$ on an interval, then f is non-decreasing on that interval
- If f is non-decreasing and differentiable, then $f'(x) \geq 0$.

3.17 Corollary. *Assume f is a differentiable function. Then the intervals where $f'(x) \geq 0$ are the intervals where $f(x)$ is increasing.*

We can define *decreasing* and *non-increasing* by reversing the inequalities we had before. f is decreasing if whenever $x_1 < x_2$ are in the domain of f, then $f(x_1) > f(x_2)$. It is non-increasing if $f(x_1) \geq f(x_2)$ holds for any two points x_1 and x_2 in the domain with $x_1 < x_2$.

Where $f(x)$ is decreasing, $-f(x)$ is increasing.

Applying the foregoing to $-f(x)$ in place of $f(x)$ we find:

3.18 Corollary. *The intervals where $f'(x) \leq 0$ are intervals where $f(x)$ is non-increasing.*

3.19 Examples.

1. Find the intervals where $f(x) = 3x^2 + 4x - 5$ is increasing and the intervals where it is decreasing.
2. Find the intervals where $f(x) = x^3 - 2x^2 + x + \frac{1}{2}$ is increasing and the intervals where it is decreasing.
3. Find the intervals where $f(x) = \dfrac{5x - 8}{(x - 2)^2}$ is increasing and the intervals where it is decreasing.

3.20 Max and min points problems. Where the problem (can be a practical problem) involves finding the largest value of a continuous function $f(x)$ on a finite closed interval $a \leq x \leq b$ and we assume $f'(x)$ exists at all interior points of the interval, the following method will work:

- Find all critical points (solutions of $f'(x) = 0$) in the interval.

  Typically just a few points $c_1, c_2, \ldots$
- Compare values $f(x)$ at the end points and at the critical points. $f(a), f(b), f(c_1), f(c_2), \ldots$.

  Largest of these is the largest value (absolute max) — and the smallest of these is the absolute min.

3.21 Examples.

(i) Find the largest value of $f(x) = 3x^4 - 8x^3 - 48x^2 + 4$ for $0 \leq x \leq 5$.

(ii) A farmer wants to fence off a rectangular paddock in the middle of a big field. He has 20 metres of fencing to use. What dimensions should the paddock be if the area is to be as large as possible?

We will now go back to something promised earlier, an account of the Mean Value Theorem and what it has to do with showing that a function that has positive derivative on an interval is an increasing function on that interval.

Recall again that a theorem is like a law of nature (or Physics or Chemistry or Biology) — some fact that is always true in a certain situation. In Science, you check out a law by checking that it is valid in many experiments. In mathematics we use a different method, of giving an incontrovertible line of reasoning to show that any time we are in a certain situation we must have the stated conclusion.

Our first step towards the Mean Value Theorem is a theorem which is fairly surprising because it seems we assume very little. The fact that we can make any conclusion about all functions or graphs that satisfy these few assumptions is maybe surprising.

3.22 Theorem (Rolle's Theorem)**.** *Suppose $y = f(x)$ is a continuous function on a finite closed interval $a \leq x \leq b$ and that $f'(x)$ exists for each x in the open interval $a < x < b$. Suppose also that $f(a) = f(b)$.*

Then there must be some point c in the open interval (that is with $a < c < b$) where $f'(c) = 0$.

If you try to picture what this says, it is that if we have a graph that starts and ends at the same level $y = f(a) = f(b)$, then the graph must have a horizontal tangent somewhere along the way.

There are some other assumptions about the graph being continuous on the closed interval and differentiable on the open interval. These are pretty mild assumptions, at least for us. Most functions we deal with are continuous and differentiable anyhow, but we do see functions with bad points here and there. What is important here is that the bad points are not included between $x = a$ and $x = b$.

You could perhaps try to convince yourself that the theorem is true by trying to draw graphs that start and end at the same level (or height y). Continuous means that you are not allowed to lift your pen while doing the drawing. Differentiable is a bit harder, but it rules out sharp corners in the graph (like an angle). Maybe if you try drawing such a graph (remember graphs can't double back on themselves) and look at what you have drawn, you will see a point with a horizontal tangent line.

You can't really convince anyone with pictures like this. Perhaps you are deliberately not drawing the kind of situation where the conclusion is false. So here is a proof, though it is based on something quite difficult that we did not prove (Theorem 3.7).

Proof. Start with any function $y = f(x)$ that satisfies the 3 assumptions (1) continuous $a \leq x \leq b$, (2) differentiable $a < x < b$, and (3) $f(a) = f(b)$.

From Theorem 3.7, we can say there is an absolute max x_M and an absolute min x_m. So these are points $a \leq x_M \leq b$ and $a \leq x_m \leq b$ which satisfy

$$f(x_m) \leq f(x) \text{ and } f(x) \leq f(x_M) \text{ for each } x \text{ with } a \leq x \leq b.$$

Now we can almost use Theorem 3.11 (which we did pretty much prove) to say that $f'(c) = 0$ for $c = x_M$. Well, not quite because x_M could be one of the endpoints $x = a$ or $x = b$ and then Theorem 3.11 won't apply to it.

However, we could equally use Theorem 3.11 on $c = x_m$ — unless that is also an end point.

But if both x_M and x_m is an end point then the largest value of $y = f(x)$ coincides with the smallest because $f(a) = f(b)$. That means we have a constant function (neither goes above not goes below $y = f(a) = f(b)$). Constant functions have derivative 0 everywhere, and so in this case we can choose any c between a and b to get $f'(c) = 0$.

What we have is a reason to see that no matter what, there has to be a c with $f'(c) = 0$. So we have proved the theorem. $\square$

The Mean Value theorem seems better than Rolle's theorem because there is no longer the need to assume $f(a) = f(b)$. But what could be say about any old graph (continuous $a \leq x \leq b$ and differentiable $a < x < b$)? We cannot say it must have a horizontal tangent because the graph could be the straight line joining the start point to the end point. The start point is where $x = a$ and $y = f(a)$. The end point is where $x = b$ and $y = f(b)$. The slope of the line joining these two points is

$$m = \frac{f(b) - f(a)}{b - a}$$

(by the formula for the slope of the line through 2 points). So for the straight line graph, the tangent always has this slope m.

The theorem says that every other graph has to have this slope somewhere.

3.23 Theorem (Mean Value Theorem (for derivatives))**.** *Suppose $y = f(x)$ is a continuous function on a finite closed interval $a \leq x \leq b$ and that $f'(x)$ exists for each x in the open interval $a < x < b$.*

Then there is some c with $a < c < b$ (so in the open interval) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Perhaps we won't go into a proof of this. You can look in the book by Anton for a proof.

In fact the Mean Value theorem includes Rolle's theorem (because if $f(a) = f(b)$ then the Mean Value Theorem says just the same as Rolle's theorem: $f'(c) = 0$).

What is maybe a surprise is that the proof of the Mean Value theorem is a little trick to reduce back to the case of Rolle's theorem. What you do is look at

$$g(x) = f(x) - m(x - a)$$

where m is the slope $(f(b) - f(a))/(b - a)$. Notice that it works out that $g(a) = f(a)$ and $g(b) = f(a)$ also. Rolle's theorem says there is a c with $g'(c) = 0$ (when we apply it to G, bot to f). Since $g'(x) = f'(x) - m$, this is pretty much the whole proof of the Mean Value Theorem.

3.24 Example. Suppose you drive from Dublin to Cork, a distance of 288km and it takes you 5 hours. Then, on average you will have travelled at $288/5 = 57.6$km/h. The Mean Value theorem says that there must have been some instant when your speed was exactly this 57.6km/h.

To see why it says that, let $x = x(t)$ be the distance you have travelled t hours from the start of your trip. Then $x(0) = 0$ (at the start you have not gone any of the distance). Assuming we measure distance in kilometres, we have $x(5) = 288$. The Mean Value theorem says there must be some c between 0 and 5 where

$$x'(c) = \frac{x(5) - x(0)}{5 - 0} = \frac{288}{5} = 57.6$$

But $x'(t)$ represents your speed at instant t. So at the instant $t = c$, the speed was 57.6 km/h.

Now, perhaps this is kind of common sense. You could (theoretically maybe) have gone an exact 57.6km/h the whole way. Or you could have gone more slowly some of the time. But if you were going more slowly for a while, you must have gone faster some other time to make up for lost average speed. And when speeding up or slowing down from less that 57.6 to more (or vice versa) there will be an instant where your speed is exactly 57.6.

Now here is the promised use of the Mean Value Theorem to prove the earlier theorem.

Proof. (of Theorem 3.15) Take x_1 and x_2 in the interval where $f'(x) > 0$ with $x_1 < x_2$. In the Mean Value theorem, take $a = x_1$ and $b = x_2$.

We have to be sure we can apply the Mean Value Theorem. We need $f(x)$ continuous $a \leq x \leq b$, but this is true because we have $f(x)$ differentiable on some interval that includes both a and b. So we have f also continuous on that interval. Then the interval $a \leq x \leq b$ is smaller. We also need $f'(x)$ to exist for $a < x < b$, but we know that because we are assuming $f'(x)$ exists on a bigger interval.

So the Mean Value theorem assures of there is some c with $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

We don't have any idea where c is, except we know $a < c < b$. But wherever it is between a and b, we know $f'(c) > 0$. So we know we have to have

$$\frac{f(b) - f(a)}{b - a} > 0$$

Remember $a = x_1$ and $b = x_2$. So what we have is actually the same as saying

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Since $x_1 < x_2$, we have $x_2 - x_1 > 0$ and therefore we know

$$f(x_2) - f(x_1) > 0,$$

which is the same as $f(x_1) < f(x_2)$.

We started with any x_1 and x_2 (in the interval) where $x_1 < x_2$. We were able to conclude that $f(x_1) < f(x_2)$ is always true. That is what it mens to say that f is increasing. $\square$

3.25 First derivative test. Aim: If $x = c$ is a critical point of $y = f(x)$, decide if it is a local max, a local min or neither.

Test:

- If $f'(x) > 0$ for all $x < c$ close to c and $f'(x) < 0$ for all $x > c$ close to c (that is if $f'(x)$ *changes sign* from $+$ to $-$ at $x = c$) then $x = c$ is a local max.

- If $f'(x)$ *changes sign* from $-$ to $+$ at $x = c$, $x = c$ is a local min.

- If $f'(x)$ has the same sign for all x on either side of c (close to c) then $x = c$ is **neither** a local max nor a local min.

3.26 Examples. (i) A farmer wants to fence off a rectangular paddock along the side of a big field. He will use existing fencing for one side of the paddock and new fencing for the other 3 sides. What dimensions should the paddock be if the area is to be 100 m^2 and the farmer wants to use as little fencing as possible?

(ii) A company wants to make milk cartons in the shape of a box. The cartons are to be one litre in volume. The distribution department insists that the length of the cartons should be $5/4$ times the width. (Ignoring folds or other complicating factors) what dimensions should the cartons be so as to minimize the surface area (corresponding to the area of cardboard used in each carton).

3.27 Sign of the second derivative. If a function $y = f(x)$ is differentiable everywhere (or for every x where it is defined) then we end up with a new function $\frac{dy}{dx} = f'(x)$, the derivative.

It is perfectly possible that this new function can also be differentiated, leading to a derivative of a derivative

$$\frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

If this makes sense it is called the *second derivative* of f and it denoted $f''(x)$ or $\frac{d^2y}{dx^2}$.

As a simple example, say $y = f(x) = 4x^3 + 2x^2 - x + 12$. Then $dy/dx = f'(x) = 12x^2 + 4x - 1$ and the derivative of that again is

$$\frac{d^2y}{dx^2} = f''(x) = 24x + 4.$$

Once we are happily able to differentiate functions given by more or less any formula, we can get a formula for the derivative, and then we can also differentiate that again to get the second derivative.

We had a graphical interpretation of the first derivative $f'(a)$ as the slope of the tangent line, but we do not have such a simple interpretation of the second derivative. However we can figure out what the sign of $f''(x)$ means graphically. At least we can if we have the same sign on an interval of x's.

Remember that $f''(x)$ is the derivative of something. It is the derivative of $f'(x)$. So, applying what we found out above, if something has a positive derivative on an interval, it means that that thing is increasing.

So, if $f''(x) = \frac{d}{dx}(f'(x)) > 0$ on an interval, we can say that $f'(x)$ is increasing on that interval. If we think about it, we can interpret this graphically. $f'(x)$ is the slope of the tangent line to the graph. So $f''(x) > 0$ on an interval means that the slope of the tangent line to the graph increases as we step from left to right in that interval.

Thinking about what it means for one line to have greater slope than another, we can say that $f'(x)$ increasing means that the tangent line twists in the anticlockwise direction as we go from left to right. In terms of the graph this means that the graph is bending towards the upwards side.

It does *not* have to do with whether the graph is actually sloping up or down, only about which way it is curving or bending.

To say $y = f(x)$ is *concave up* on an interval, means that the slope of the graph is increasing on that interval.

A more geometric way to express it is this: all chords of the graph lie above the graph.

Concave down is the opposite.

3.28 Theorem. *Provided we restrict to f where $f''(x)$ exists, the intervals where $f''(x) \geq 0$ are the intervals where the graph $y = f(x)$ is concave up.*

The intervals where $f''(x) \leq 0$ are the intervals where the graph $y = f(x)$ is concave down.

3.29 Definition. A point $x = c$ on a graph $y = f(x)$ where the graph changes from concave up to concave down (or vice versa, from concave down to concave up) is called a *point of inflection*.

In practice these are points where $f''(x)$ changes sign.

3.30 Example. For the graph $y = x^4 - 2x^3 - 120x^2 + 5x - 11$, find the intervals where the graph is concave upwards and the intervals where it is concave downwards.

We need to examine the sign of d^2y/dx^2 and so we need to work it out first.

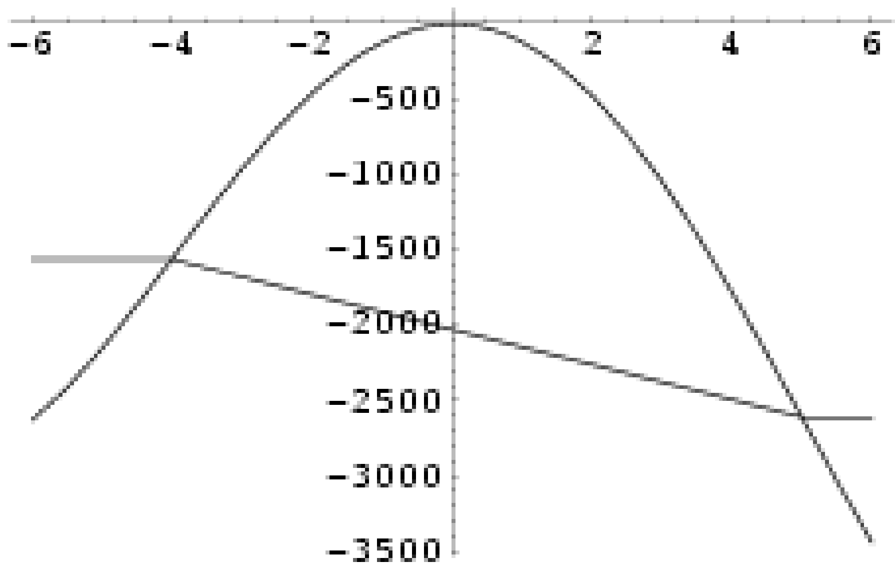
$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 6x^2 - 240x + 5 \\ \frac{d^2y}{dx^2} &= 12x^2 - 12x - 240 \\ &= 12(x^2 - x - 20) \\ &= 12(x-5)(x+4) \end{aligned}$$

We can work out the sign of this if we know the sign of each factor. The factors have a chance to change sign when one of them is 0, that is at $x = -4$ or $x = 5$.

	$x < -4$	$-4 < x < 5$	$5 < x$
$x + 4$	$-$	$+$	$+$
$x - 5$	$-$	$-$	$+$
$\frac{d^2y}{dx^2} = 12(x-5)(x+4)$	$+$	$-$	$+$
y	concave up	concave down	concave up

In summary the graph is concave up on the interval $(-\infty, -4]$ and again on the interval $[5, \infty)$. It is concave down on $[-4, 5]$.

In the graph shown below, you can certainly see the concave down section fairly clearly, but the concave up sections look nearly straight.



3.31 Second derivative test. Aim: Decide if a critical point $x = c$ (i.e. $f'(c) = 0$) is a local max or min.

Test: (Technically we need $f''(x)$ to exist and be continuous at c for this test.)

- If $f''(c) > 0$ (note: consequently the graph is concave up near c as well as having a horizontal tangent line) then $x = c$ is a *local min*
- If $f''(c) < 0$ then $x = c$ is a *local max*
- If $f''(c) = 0$ then *no conclusion* can be made (without further information)

  Examples to show this: $f(x) = x^3$ and $f(x) = x^4$ and $c = 0$. (In both cases $f'(c) = 0$ [so we are at a critical point] and also $f''(c) = 0$ [so we are in the case where the second derivative test fails]. In the case of $y = x^3$ there is neither a relative maximum nor a relative minimum, while in the case $y = x^4$ there is a relative minimum at $x = 0$. In the case $y = x^4$, the second derivative $d^2y/dx^2 = 12x^2$ does *not* change sign at $x = 0$ and so 0 is *not* a point of inflection.)

3.32 Graphing. To make a graph of a function $y = f(x)$ that is "true to the function" we should include the main features of the graph (or we could focus deliberately on one part of the graph if our aim is to highlight one feature).

The overall features to show would normally be:

- points where the graph crosses either axis ($x = 0$ and $y = 0$)
- critical points (points where $f'(x) = 0$)

  These will include the local max and local min points.
- points of inflection ($f''(x) = 0$ and changes sign)

- long run behaviour of the graph if it can be indicated by *asymptotes*, a concept which we will now explain.

3.33 Asymptotes. Asymptotes are straight lines that indicate the long run behaviour of certain graphs. They are not actually part of the graph but they can help describe graphs.

Asymptotes can be vertical, horizontal or oblique (means nonzero slope) lines.

3.34 Vertical Asymptotes. A graph $y = f(x)$ has the line $x = a$ as a vertical asymptote if either

$$\lim_{x \to a^+} |f(x)| = \infty \text{ or } \lim_{x \to a^-} |f(x)| = \infty$$

Usually they occur where a denominator of $y = f(x)$ is zero (the only catch is that sometimes the formula can be simplified so that the denominator is not zero there — if that is going to happen then the original numerator is zero as well as the denominator).

3.35 Examples.

1. $y = \frac{x}{x-1}$

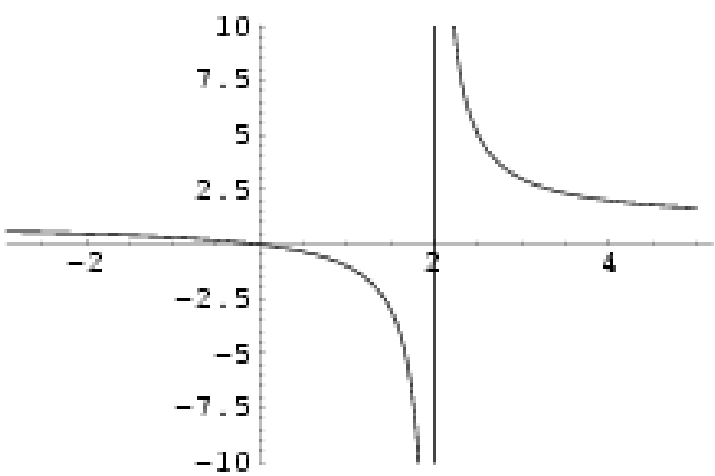
   Vertical asymptote at $x = 1$

2. $y = \frac{x^2-x}{x^2-3x+2}$

   Seems to have vertical asymptotes at $x = 1$ and $x = 2$, but the numerator $x^2 - x$ is also 0 at $x = 1$. This means we can simplify

   $$y = \frac{x^2 - x}{x^2 - 3x + 2} = \frac{x(x-1)}{(x-1)(x-2)} = \frac{x}{x-2}$$

   and only $x = 2$ is a vertical asymptote. Here is the graph with the vertical asymptote.



3.36 Horizontal Asymptotes. A graph $y = f(x)$ has the horizontal line $y = y_0$ as an asymptote if

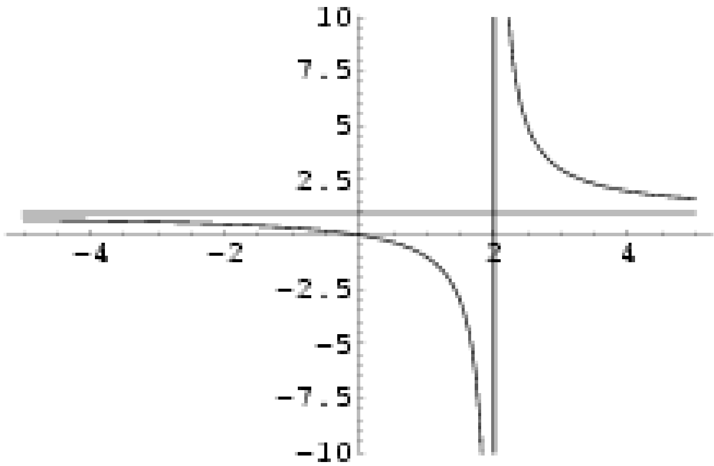
$$\lim_{x \to +\infty} f(x) = y_0 \text{ or } \lim_{x \to -\infty} f(x) = y_0$$

3.37 Example. $y = \frac{x}{x-2}$

Here

$$\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty} \frac{1}{1-2/x} = 1$$

and so the line $y = 1$ is an asymptote (which is horizontal). Here is the graph with both the horizontal asymptote and the vertical one.



3.38 Oblique Asymptotes. A graph $y = f(x)$ has the (sloping or oblique) line $y = mx + c$ as an asymptote if

$$\lim_{x\to+\infty} f(x) - (mx+c) = 0 \text{ or } \lim_{x\to-\infty} f(x) - (mx+c) = 0$$

These usually happen where the degree of the numerator (meaning, the highest power of x present after multiplying out and simplifying) is one higher than the degree of the denominator.

Notice that the idea is that the graph $y = f(x)$ follows the line very closely in the far distance — as was the case with the other kinds of asymptotes.

3.39 Example. $y = \frac{3x^2+x-2}{x+2}$.

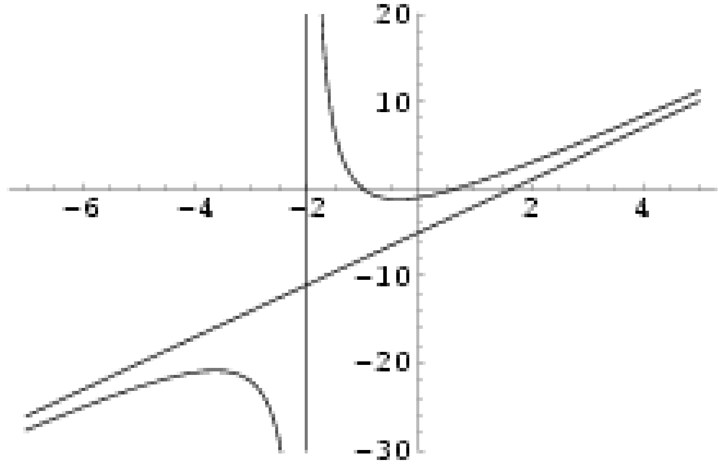
Looking at highest powers in numerator and denominator, which will outweigh the other terms for $|x|$ large, we see that y is roughly $\frac{3x^2}{x} = 3x$. Check

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{3x^2+x-2}{x+2} - 3x &= \lim_{x\to\pm\infty} \frac{-5x-2}{x+2} \\
&= \lim_{x\to\pm\infty} \frac{-5-2/x}{1+2/x} \\
&= -5
\end{aligned}$$

We see then that if we subtract the -5,

$$\lim_{x\to\pm\infty} \frac{3x^2+x-2}{x+2} - (3x-5) = 0$$

and the asymptote is $y = 3x - 5$.



3.40 Symmetry. Sometimes graphs have a symmetry that can help with graphing.

For example if $y = f(x)$ is an *even function* (meaning $f(-x) = f(x)$) then the graph is symmetrical in the y-axis.

Examples of even graphs are $y = 16x^4 - 3x^2 + 121$ (only even powers) or $y = \cos x$.

For *odd functions* (means $f(-x) = -f(x)$) the graph $y = f(x)$ will be symmetrical in the origin.

Examples are $y = 21x^5 + 16x^3 - 15x$ (only odd powers) $y = \sin x$ ($\sin(-x) = -\sin x$ is true for every x) and $y = \tan x$.