1S2 (Timoney) Tutorial sheet 4
[November 14 – 19, 2007]

Name: Solutions

1. For $\mathbf{v} = -3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = 6\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$, calculate

   (a) the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$
   Solution: We know $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Since $\cos\theta$ is what we are looking for, what we need is to compute the other quantities in the equation

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (-3)(6) + 7(-3) + (2)(7) \\ &= -18 - 21 + 14 = -25 \\ \|\mathbf{v}\| &= \sqrt{(-3)^2 + 7^2 + 2^2} = \sqrt{62} \\ \|\mathbf{w}\| &= \sqrt{6^2 + (-3)^2 + 7^2} = \sqrt{94} \end{aligned}$$

   So we know $-25 = \sqrt{62}\sqrt{94}\cos\theta$ and so the answer is

$$\cos\theta = \frac{-25}{\sqrt{62}\sqrt{94}} = \frac{-25}{2\sqrt{1457}}.$$

   (b) The projection $\mathrm{proj}_{\mathbf{w}}(\mathbf{v})$ of $\mathbf{v}$ along the direction of $\mathbf{w}$.
   Solution:

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{-25}{94}\mathbf{w} = \left(\frac{-150}{94}\right)\mathbf{i} + \left(\frac{75}{94}\right)\mathbf{j} + \left(\frac{-175}{94}\right)\mathbf{k}$$

   (c) the unit vector in the same direction as $\mathbf{w}$
   Solution: The unit vector is $\mathbf{w}$ divided by its length $\|\mathbf{w}\|$, that is

$$\frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{94}}\mathbf{w} = \frac{6}{\sqrt{94}}\mathbf{i} - \frac{3}{\sqrt{94}}\mathbf{j} + \frac{7}{\sqrt{94}}\mathbf{k}$$

2. Find the equation of the plane in space passing through the point $(1, 2, 3)$ perpendicular to the vector $6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$

   Solution: We know the equation has the form $6x - 5y + 4z = \mathrm{const}$ (the coefficients of x, y and z are the compnents of the normal vector). Since $(1, 2, 3)$ is one point on the plane, we must have

$$6 - 10 + 12 = \mathrm{const}$$

   and so the equation is $6x - 5y + 4z = 8$.

3. Find the equation of the plane (in space) passing through the points $(2, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 8)$.

   Solution: We know the equation has the form

$$ax + by + cz = d$$

   for some constants a, b, c and d (with a, b, c not all 0). Plugging in the 3 points we find

$$\begin{aligned} 2a &= d \\ 5b &= d \\ 8c &= d \end{aligned}$$

   So if we take $d = 1$ we find $a = 1/2$, $b = 1/5$ and $c = 1/8$. This gives us an equation

$$\frac{1}{2}x + \frac{1}{5}y + \frac{1}{8}z = 1$$

   That's a perfectly good answer but you might prefer to multiply it across by 40 to get the equivalent equation

$$20x + 8y + 5z = 40$$

   (which has no fractions).

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