

1S2 (Timoney) Tutorial sheet 20

[April 28 – May 2, 2008]

Name: Solutions

1. A loaded die has the following probabilities of showing the numbers 1–6 after a throw:

$$\frac{3}{15}, \frac{2}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}, \frac{4}{15}$$

(in that order). Find the probability that a number ≥ 4 will show after the die is thrown.

Solution:

$$P(4) + P(5) + P(6) = \frac{2}{15} + \frac{1}{15} + \frac{4}{15} = \frac{7}{15}$$

2. A random variable X associated with the outcome of tossing the same die as in the previous question has the values

$$X(1) = X(3) = 1, X(2) = X(6) = -2, X(4) = X(5) = 4.$$

Find the mean of the random variable.

Solution:

$$\begin{aligned}\mu &= P(1)X(1) + P(2)X(2) + \cdots + P(6)X(6) \\ &= \frac{3}{15}1 + \frac{2}{15}(-2) + \frac{3}{15}1 + \frac{2}{15}4 + \frac{1}{15}4 + \frac{4}{15}(-2) \\ &= \frac{1}{15}(3 - 4 + 3 + 8 + 4 - 8) \\ &= \frac{6}{15}\end{aligned}$$

3. A loaded coin has probability $p = 0.495$ of coming up heads. If the coin is tossed 11 times, find the probability that the number of heads will be either 5 or 6.

Solution: This is an application of the binomial distribution ($n = 11$ independent trials, probability of success in each trial $= p = 0.495$ and probability of failure $= q = 1 - p = 0.505$). What we want is

$$P(5) + P(6) = \binom{11}{5} p^5 q^{11-5} + \binom{11}{6} p^6 q^{11-6} = \binom{11}{5} p^5 q^6 + \binom{11}{6} p^6 q^5$$

We have

$$\binom{11}{5} = \frac{11!}{5!6!} = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} = 462$$

and $\binom{11}{6} = 462$ also. $p^5 q^6 = 0.000492918$, $p^6 q^5 = 0.000483157$ and the answer turns out to be 0.450946.

4. For the same coin as in the previous question, in the long run, what is the expected number of heads that would show up for the experiment of tossing the coin 11 times? (This is also known as the [theoretical] mean.) And what is the (theoretical) variance for the experiment?

Solution: $\mu = np = 11 \times 0.495 = 5.445$ is the mean.

$\sigma^2 = npq = 11 \times 0.495 \times 0.505 = 2.74973$ is the variance.

5. The number of alpha particles hitting a certain detector per second is found to obey a Poisson distribution with mean 0.4. What then is the probability of observing 0, 1 or 2 alpha particles in a given second?

Solution: The Poisson distribution gives probability

$$P(i) = e^{-\mu} \frac{\mu^i}{i!}$$

to each of the numbers $i = 0, 1, 2, \dots$. Here we want $\mu = 0.4$ and the sum

$$P(0) + P(1) + P(2) = e^{-0.4} \left(1 + 0.4 + \frac{(0.4)^2}{2!} \right) = 0.992074.$$

Extra Problem for next week. A factory produces bottles of a soft drink that are sold as 2 litre bottles. A good model is that the quantity of liquid in a bottle obeys a normal distribution with mean 2.02 (litres) and standard deviation 0.09. What proportion of the bottles have less than 2 litres in them?

Solution: We are supposing that the probability of $< x$ (litres in a bottle) is given by a normal distribution function

$$F_{\mu,\sigma}(x) = F_{2.02,0.09}(x) = F_{0,1}\left(\frac{x-\mu}{\sigma}\right) = F_{0,1}\left(\frac{x-2.02}{0.09}\right)$$

(in terms of the standard normal distribution $F_{0,1}$ to be found in the tables). What we want is

$$F_{2.02,0.09}(2) = F_{0,1}\left(\frac{2-2.02}{0.09}\right) = F_{0,1}(-0.222222) = 1 - F_{0,1}(0.222)$$

(by symmetry of the standard normal). From the tables, $F_{0,1}(0.22) = 0.5871$ while $F_{0,1}(0.23) = 0.5910$ and we could guess that 0.222 would add about 2/10 of the difference, or about 0.0008, giving as our answer

$$1 - 0.5879 = 0.4121$$

So about this fraction of the bottles (or 41.21% of them) can be expected to have < 2 litres in them.

Richard M. Timoney