1S2 (Timoney) Tutorial/exercise sheet 1
[October 22– 26, 2007]

Name: Solutions

1. Write an augmented matrix corresponding to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 5x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & -3 \\ 2x_1 & + & 3x_2 & + & 7x_3 & + & 2x_4 & = & 18 \\ -x_1 & - & 12x_2 & - & 11x_3 & - & 16x_4 & = & -37 \\ x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & -3 \end{array}$$

Solution:

$$\left[\begin{array}{cccccr} 5 & -2 & 1 & -4 & : & -3 \\ 2 & 3 & 7 & 2 & : & 18 \\ -1 & -12 & -11 & -16 & : & -37 \\ 1 & 2 & -1 & -1 & : & -3 \end{array}\right]$$

2. Write out a system of linear equations (in the unknowns x_1, x_2 and x_3) corresponding to the augmented matrix:

$$\left[\begin{array}{rcrcc} 1 & 0 & -1 & : & 5 \\ -2 & 1 & 4 & : & 3 \\ 0 & 5 & 2 & : & 1 \end{array}\right]$$

Solution:

$$\begin{array}{rcrcrcl} x_1 & & & - & x_3 & = & 5 \\ -2x_1 & + & x_2 & + & 4x_3 & = & 3 \\ & & 5x_2 & + & 2x_3 & = & 1 \end{array}$$

3. Use Gaussian elimination followed by back-substitution to solve the system of equations corresponding to the augmented matrix of the previous question.

Solution:

$$\left[\begin{array}{ccrcr} 1 & 0 & -1 & : & 5 \\ 0 & 1 & 2 & : & 13 \\ 0 & 5 & 2 & : & 1 \end{array}\right] \begin{array}{l} \\ \text{OldRow2} + 2 \times \text{OldRow1} \\ \\ \end{array}$$

$$\left[\begin{array}{ccrcr} 1 & 0 & -1 & : & 5 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & -8 & : & -64 \end{array}\right] \begin{array}{l} \\ \\ \text{OldRow3} - 5 \times \text{OldRow2} \end{array}$$

$$\left[\begin{array}{ccrcr} 1 & 0 & -1 & : & 5 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 1 & : & 8 \end{array}\right] \begin{array}{l} \\ \\ \text{OldRow3} \times \frac{-1}{8} \end{array}$$

Translate to equations now, and do back substitution.

$$\begin{cases} x_1 & & - & x_3 & = & 5 \\ & x_2 & + & 2x_3 & = & 13 \\ & & & x_3 & = & 8 \end{cases}$$

$$\begin{cases} x_1 & = & 5 + x_3 \\ x_2 & = & 13 - 2x_3 \\ x_3 & = & 8 \end{cases}$$

$$\begin{cases} x_1 & = & 5 + 8 = 13 \\ x_2 & = & 13 - 2x_3 = 13 - 16 = -3 \\ x_3 & = & 8 \end{cases}$$

Thus we end up with the one solution $x_1 = 13$, $x_2 = -3$ and $x_3 = 8$.

4. Use Gauss-Jordan elimination to solve the system of equations corresponding to same augmented matrix (of the previous two questions).

   Solution: Proceed as before to reach the stage

$$\left[\begin{array}{cccc} 1 & 0 & -1 & : & 5 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 1 & : & 8 \end{array}\right| \begin{array}{l} \\ \\ \text{OldRow3} \times \frac{-1}{8} \end{array}$$

   and now

$$\left[\begin{array}{cccc} 1 & 0 & 0 & : & 13 \\ 0 & 1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 8 \end{array}\right| \begin{array}{l} \text{OldRow1} + \text{OldRow3} \\ \text{OldRow2} - 2 \times \text{OldRow3} \\ \\ \end{array}$$

   Translating to equations now, we get

$$\begin{cases} x_1 & = & 13 \\ x_2 & = & -3 \\ x_3 & = & 8 \end{cases}$$

   (of course, the same result as last time).

5. Use Gauss-Jordan elimination to describe all solutions of the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 5x_1 & - & x_2 & - & x_3 & - & 8x_4 & = & 5 \\ 10x_1 & - & 2x_2 & + & 4x_3 & + & 2x_4 & = & 8 \end{array}$$

   Solution: The augmented matrix is

$$\left[\begin{array}{cccccc} 5 & -1 & -1 & -8 & : & 5 \\ 10 & -2 & 4 & 2 & : & 8 \end{array}\right]$$

$$\left[\begin{array}{cccccc|} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 \\ 10 & -2 & 4 & 2 & : & 8 \end{array}\right. \text{OldRow1} \times \frac{1}{5}$$

$$\left[\begin{array}{cccccc|l} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 & \\ 0 & 0 & 6 & 18 & : & -2 & \text{OldRow2} - 10 \times \text{OldRow1} \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 & \\ 0 & 0 & 1 & 3 & : & -\frac{1}{3} & \text{OldRow2} \times \frac{1}{6} \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & -\frac{1}{5} & 0 & -1 & : & \frac{14}{15} & \text{OldRow1} + \frac{1}{5} \times \text{OldRow2} \\ 0 & 0 & 1 & 3 & : & -\frac{1}{3} & \end{array}\right.$$

Writing as equations we get

$$\left\{\begin{array}{ccccccc} x_1 & - & \frac{1}{5}x_2 & & - & x_4 & = & \frac{14}{15} \\ & & & x_3 & + & 3x_4 & = & -\frac{1}{3} \end{array}\right.$$

or

$$\left\{\begin{array}{ccl} x_1 & = & \frac{14}{15} + \frac{1}{5}x_2 + x_4 \\ x_3 & = & -\frac{1}{3} - 3x_4 \\ & & x_2, x_4 \text{ free} \end{array}\right.$$

(solved for x_1 and x_2 in terms of the free variables x_2 and x_4).

6. Find all solutions of the following system of linear equations by using Gauss-Jordan elimination.

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & -3 \\ x_1 & + & 3x_2 & + & 7x_3 & + & 2x_4 & = & 18 \\ x_1 & - & 12x_2 & - & 11x_3 & - & 16x_4 & = & -37 \end{array}$$

Solution: The augmented matrix is

$$\left[\begin{array}{cccccc} 1 & -2 & 1 & -4 & : & -3 \\ 1 & 3 & 7 & 2 & : & 18 \\ 1 & -12 & -11 & -16 & : & -37 \end{array}\right]$$

$$\left[\begin{array}{cccccc|l} 1 & -2 & 1 & -4 & : & -3 & \\ 0 & 5 & 6 & 6 & : & 21 & \text{OldRow2} - \text{OldRow1} \\ 0 & -10 & -12 & -12 & : & -34 & \text{OldRow3} - \text{OldRow1} \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & -2 & 1 & -4 & : & -3 & \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & : & \frac{21}{5} & \text{OldRow2} \times \frac{1}{5} \\ 0 & -10 & -12 & -12 & : & -34 & \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & -2 & 1 & -4 & : & -3 & \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & : & \frac{21}{5} & \\ 0 & 0 & 0 & 0 & : & 8 & \text{OldRow3} + 10 \times \text{OldRow2} \end{array}\right.$$

To follow Gauss-Jordan to the letter, we should continue further. But the last row now corresponds to the equation $0 = 8$ and so we see that there are no solutions to this system. Hence no solutions to the original system.

For the sake of practice we finish off the Gauss-Jordan method.

$$\left[\begin{array}{cccccc|l} 1 & -2 & 1 & -4 & : & -3 & \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & : & \frac{21}{5} & \\ 0 & 0 & 0 & 0 & : & 1 & \text{OldRow3} + \times\frac{1}{8} \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & -2 & 1 & -4 & : & 0 & \text{OldRow1} + 3 \times \text{OldRow3} \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & : & 0 & \text{OldRow1} - \frac{21}{5} \times \text{OldRow3} \\ 0 & 0 & 0 & 0 & : & 1 & \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & 0 & \frac{17}{5} & -\frac{8}{5} & : & 0 & \text{OldRow1} + 2 \times \text{OldRow2} \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & : & 0 & \\ 0 & 0 & 0 & 0 & : & 1 & \end{array}\right.$$

System of equations for this augmented matrix is:

$$\left\{ \begin{array}{rrrrrrrrr} x_1 & & + & \frac{17}{5}x_3 & - & \frac{8}{5}x_4 & = & 0 \\ & x_2 & + & \frac{6}{5}x_3 & + & \frac{6}{5}x_4 & = & 0 \\ & & & & & 0 & = & 1 \end{array} \right.$$

We can see (again) that these are inconsistent. There are no solutions to the system.

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