1S2 (Timoney) Tutorial sheet 19
[April 21–25, 2008]

Name: Solutions

1. Find the equation of the line that is the best least squares fit to the data points $(2,3)$, $(3,2)$, $(5,1)$, $(6,0)$.

   Solution: Taking the desired line as $y = \beta_0 + \beta_1 x$,

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

   we would ideally want $\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ so that $X\boldsymbol{\beta}$ is the same as $\mathbf{y}$. However that is not possible and we solve the normal equations

$$X^t X \boldsymbol{\beta} = X^t \mathbf{y}$$

   We calculate

$$X^t X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}, \qquad X^t \mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

   We can solve

$$\begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

   by row reducing

$$\left[\begin{array}{ccc} 4 & 16 & : 6 \\ 16 & 74 & : 17 \end{array} \right]$$

   Divide first row by 4:

$$\left[\begin{array}{ccc} 1 & 4 & : 3/2 \\ 16 & 74 & : 17 \end{array} \right]$$

   Subtract 16 times row 1 from row 2:

$$\left[\begin{array}{ccc} 1 & 4 & : 3/2 \\ 0 & 10 & : -7 \end{array} \right]$$

   Divide row 2 by 10:

$$\left[\begin{array}{ccc} 1 & 4 & : 3/2 \\ 0 & 1 & : -7/10 \end{array} \right]$$

Subtract 4 times row 2 from row 1:

$$\left[\begin{array}{cc:c} 1 & 0 & 43/10 \\ 0 & 1 & -7/10 \end{array}\right]$$

So $\beta_0 = 43/10$ and $\beta_1 = -7/10$.

The line is $y = (43/10) - (7/10)x$.

2. Find all the solutions (also known as the general solution) of the system of differential equations

$$\begin{cases} \dfrac{dy_1}{dx} &= 19y_1 - 24y_2 \\ \dfrac{dy_2}{dx} &= 20y_1 - 33y_2 \end{cases}$$

Solution: We know that the solutions are given in terms of the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 19 & -24 \\ 20 & -33 \end{bmatrix}$$

If λ_1 and λ_2 are the eigenvalues of A and $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for those eigenvalues, then the general solution is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 e^{\lambda_1 x}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 x}\mathbf{v}_2$$

where α_1 and α_2 are arbitrary constants. (At least this is so if A has two different real eigenvalues.)

To get the eigenvalues for A we solve the characteristic equation $\det(A - \lambda I_2) = 0$. We find

$$A - \lambda I_2 = \begin{bmatrix} 19 & -24 \\ 20 & -33 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 19-\lambda & -24 \\ 20 & -33-\lambda \end{bmatrix}$$

and so

$$\begin{aligned} \det(A - \lambda I_2) &= (19-\lambda)(-33-\lambda) + 24(20) = (\lambda - 19)(\lambda + 33) + 480 \\ &= \lambda^2 + 14\lambda - 19(33) + 480 = \lambda^2 + 14\lambda - 147 \end{aligned}$$

Since $147 = 7(21)$ we can factor this as $(\lambda + 21)(\lambda - 7)$ and so the eigenvalues are $\lambda_1 = -21$ and $\lambda_2 = 7$.

To find the eigenvector $\mathbf{v}_1$ for $\lambda_1 = -21$ we row reduce $[A - \lambda_1 I_2 : \mathbf{0}]$, which is

$$\left[\begin{array}{cc:c} 40 & -24 & 0 \\ 20 & -12 & 0 \end{array}\right]$$

Divide the first row by 40:

$$\left[\begin{array}{cc:c} 1 & -3/5 & 0 \\ 20 & -12 & 0 \end{array}\right]$$

Subtract 20 times row 1 from row 2 to get a new row 2:

$$\begin{bmatrix} 1 & -3/5 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

Taking the second component (the free variable) to be 1, the first must be $3/5$ and we get the eigenvector $\begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$.

We could also take 5 times that to get $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Now for $\lambda_2 = 7$, we should row reduce

$$\begin{bmatrix} 12 & -24 & :0 \\ 20 & -40 & :0 \end{bmatrix}$$

Divide the first row by 12:

$$\begin{bmatrix} 1 & -2 & :0 \\ 20 & -40 & :0 \end{bmatrix}$$

Subtract 20 times row 1 from row 2:

$$\begin{bmatrix} 1 & -2 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

If we take the second variable as the free variable and give it value 1, we get the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So the solutions to the system are given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 e^{-21x} \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} + \alpha_2 e^{7x} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(with α_1 and α_2 arbitrary constants). We can write the same result without matrix notation as

$$\begin{cases} y_1 = (3/5)\alpha_1 e^{-21x} + 2\alpha_2 e^{7x} \\ y_2 = \alpha_1 e^{-21x} + \alpha_2 e^{7x} \end{cases}$$

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